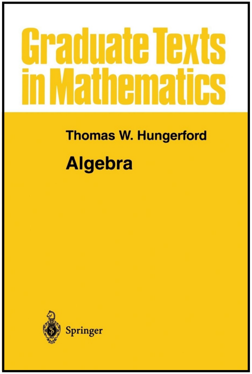
Modern Algebra

Chapter 0. Introduction: Prerequisites and Preliminaries

0.1–0.6. Logic, Sets and Classes, Functions, Relations and Partitions, Products, Integers—Proofs of Theorems



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Proof. We have

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Corollary 0.3.B

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$$f \text{ is bijective } \iff f \text{ has a two-sided inverse.}$$

Proof. If f has a left inverse then by Theorem 0.3.1(i) f is one to one. If f has a right inverse then by Theorem 0.3.1(ii) f is onto. So if f has a two-sided inverse then f is a bijection.

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Lemma 0.4.A

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Proof. If $\overline{a} \cap \overline{b} = \varnothing$ then there is $c \in \overline{a} \cap \overline{b}$. Hence $c \sim a$ and $c \sim b$. By symmetry and transitivity, $a \sim b$. So $a \in \overline{b}$. Similarly, for any $d \sim a$ we have $d \sim b$ and so $d \in \overline{b}$, or more generally $\overline{a} \subset \overline{b}$. For any $e \sim b$ we likewise have $e \in \overline{a}$ and so $\overline{b} \subset \overline{a}$, and $\overline{a} = \overline{b}$. □

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