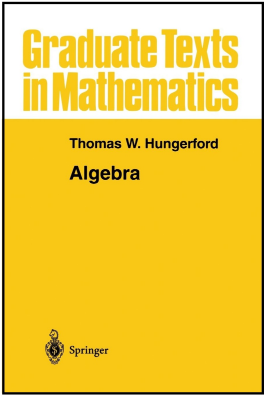
Modern Algebra

Chapter I. Groups

I.7. Categories: Products, Coproducts, and Free Objects
–Proofs of Theorems



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Theorem I.7.3

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Proof. Since P is a product, Q is an object in $\mathcal{C}$, and $\psi_i : Q \to A_i$ there is a unique morphism $g : Q \to P$ such that $\pi_i \circ g = \psi_i$ for all $i \in I$. Similarly, since Q is a product, P is an object in $\mathcal{C}$, and $\pi_i : P \to A_i$, there is a unique morphism $f : P \to Q$ such that $\psi_i \circ f = \pi_i$ for all $i \in I$. So these diagrams commute:

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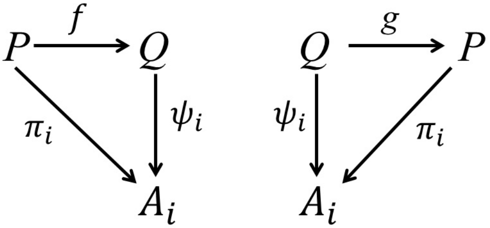

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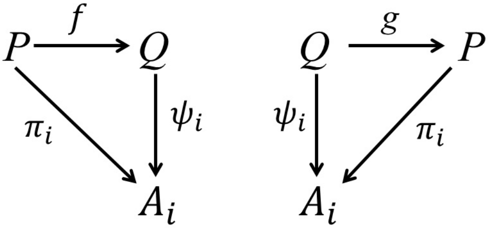

Theorem I.7.3 (continued 1)

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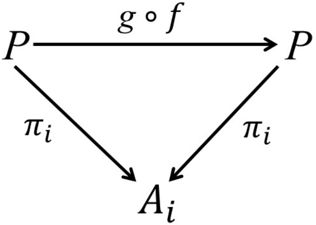
Proof (continued). So we can compose f and g to get:



So $g \circ f : P \to P$ is a morphism such that $\pi_i \circ (g \circ f) = \pi_i$ for all $i \in I$. But with $(P, \{\pi_i\})$ as a product, P as an object, and $\pi_i : P \to A_i$, there is a unique morphism mapping P to P. Since the identity 1_P is such a morphism then it must be that $g \circ f = 1_P$.

Theorem I.7.3 (continued 2)

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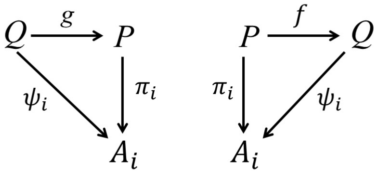
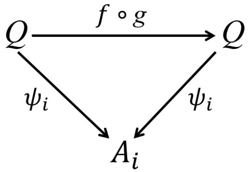
Proof (continued). Similarly we have



and



and that $f \circ g = 1_Q$. So f (and g) are equivalences and P is equivalent to Q. □

Theorem I.7.8

Theorem I.7.8. If $\mathcal{C}$ is a concrete category, if F and F' are objects of $\mathcal{C}$ such that F is free on the set X and F' is free on the set X' and $|X| = |X'|$, then F is equivalent to F'.

Proof. Since $|X| = |X'|$, there is a bijection $f : X \to X'$. Since F is free on X, there is a set map $i : X \to F$. Since F' is free on X', there is a set map $j : X' \to F'$. Consider $j \circ f : X \to F'$. Since object F is free on set X and F' is an object then there is a unique morphism $\varphi : F \to F'$ such that $j \circ f = \varphi \circ i$:

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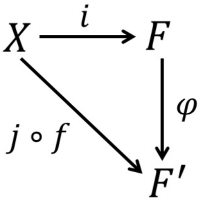

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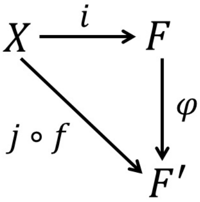

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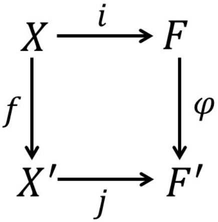
Proof (continued). Since $j : X' \to F'$ and $f : X \to X'$ we can expand this to the commutative diagram:



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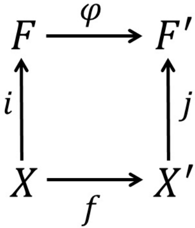

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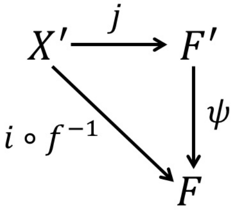
Proof (continued). Similarly, since $f : X \to X'$ is a bijection, we have $f^{-1} : X \to X$. Consider $i \circ f^{-1} : X' \to F$. Since object F' is free on set X' and F is an object then there is a unique morphism $\psi : F' \to F$ such that $i \circ f^{-1} = \psi \circ j$:



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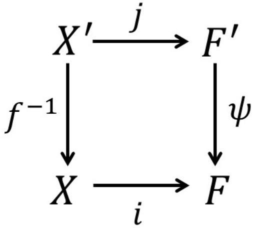

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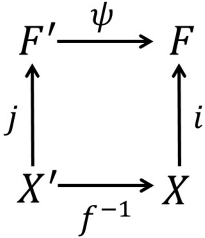

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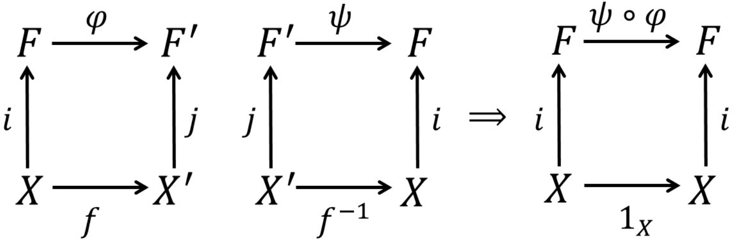
Proof (continued). Combining the above diagrams gives the commutative diagram:



Hence $(\psi \circ \varphi) \circ i = i \circ 1_X = i$. Now we also have $1_F \circ i = i$ (taking 1_F in place of $\psi \circ \varphi$). Now we had ψ and φ unique above, so we must have $\psi \circ \varphi = 1_F$. Similarly (combining the two diagrams in the opposite order) we have $\varphi \circ \psi = 1_{F'}$. Therefore (by definition) F is equivalent to F'. □

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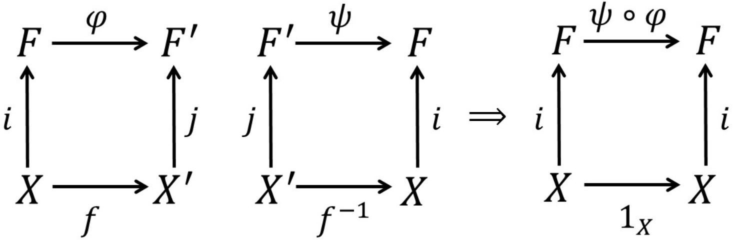

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Theorem I.7.10

Theorem I.7.10. Any two universal (respectively, couniversal) objects in a category $\mathcal{C}$ are equivalent.

Proof. Let I and J be universal objects in $\mathcal{C}$. Since I is universal and J is an object, there is a unique morphism $f : I \to J$. Similarly, since J is universal and I is an object there is a unique morphism $g : J \to I$. The composition $g \circ f : I \to I$ is a morphism of $\mathcal{C}$. But $1_I : I \to I$ is also a morphism of $\mathcal{C}$.

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