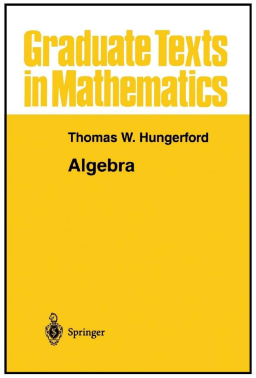
Modern Algebra

Chapter I. Groups

I.8. Direct Products and Direct Sums—Proofs of Theorems



Proposition I.8.2

Theorem I.8.2. Let $\{G_i \mid i \in I\}$ be a family of groups, let H be a group, and let $\{\varphi_i : H \to G_i \mid i \in I\}$ a family of group homomorphisms. Then there is a unique homomorphism $\varphi : H \to \prod G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$ and this property determines $\prod G_i$ uniquely up to isomorphism. (In other words, $\prod G_i$ is a product in the category of groups.)

Proof. By Theorem 0.5.2, the map of sets $\varphi : H \to \prod_{i \in I} G_i$ given by $\varphi(a) = \{\varphi_i(a)\}_{i \in I} \in \prod_{i \in I} G_i$ is the unique function such that $\pi_i \varphi = \varphi_i$ for all $i \in I$. We now confirm that φ is a homomorphism. Let $a, b \in H$. Then

$$\begin{aligned} \varphi(ab) &= \{\varphi_i(ab)\}_{i \in I} \\ &= \{\varphi_i(a)\varphi_i(b)\}_{i \in I} \text{ since } \varphi_i \text{ is a homomorphism} \\ &= \varphi(a)\varphi(b). \end{aligned}$$

Proposition I.8.2 (continued)

Proof (continued). We now consider Definition I.7.2. With $P = \prod_{i \in I} G_i$ and $\{\pi_i\}_{i \in I}$ a family of morphisms (onto homomorphisms by Theorem I.8.1(ii)), with $B = H$ any object (i.e., group) and $\{\varphi_i\}_{i \in I}$ a family of morphisms (group homomorphisms) mapping $B \to A_i$ (or $H \to G_i$ here), we have the unique morphism φ mapping $B \to P$ (i.e., unique homomorphism $\varphi : H \to \prod G_i$) such that $\pi_i \circ \varphi = \varphi_i$ for all $i \in I$. So $\prod_{i \in I} G_i$ is a product in the categorical sense as defined in Definition I.7.2 and we can use properties of categories from Section I.7.

By Theorem I.7.3, any two products of $\{G_i\}_{i \in I}$ are equivalent. That is, there are morphisms (group homomorphisms) between the two products which compose to give an identity mapping and hence the products are isomorphic. □

Proposition I.8.4(i)

Theorem I.8.4(i). If $\{G_i \mid i \in I\}$ is a family of groups, then:
(i) $\prod_{i \in I}^{w} G_i$ is a normal subgroup of $\prod_{i \in I} G_i$.

Proof. We use Theorem I.5.1(iv) and show for all $a \in \prod_{i \in I} G_i$ and $N = \prod_{i \in I}^{w} G_i$ that $aNa^{-1} \subseteq N$. Let $n \in \prod_{i \in I}^{w} G_i$. Then $n = \{n(i)\}_{i \in I}$ where $n(i) \in G_i$ and $n(i) = e_i$ for all but finitely many $i \in I$ (where e_i is the identity in G_i; say $n(i) \neq e_i$ for $i \in I_0$). Let $a \in \prod_{i \in I} G_i$. Then $a = \{a(i)\}_{i \in I}$ where $a(i) \in G_i$. So $a^{-1} = \{(a(i))^{-1}\}_{i \in I}$. Now $ana^{-1} = \{a(i)\}_{i \in I}\{n(i)\}_{i \in I}\{(a(i))^{-1}\}_{i \in I} = \{a(i)n(i)(a(i))^{-1}\}_{i \in I}$. Since $a(i), n(i), (a(i))^{-1} \in G_i$ for all $i \in I$, then $a(i)n(i)(a(i))^{-1} \in G_i$ for all $i \in I$. Since $n(i) = e_i$ for all $i \in I \setminus I_0$ then $a(i)n(i)(a(i))^{-1} = a(i)e_i(a(i))^{-1} = e_i$ for all $i \in I \setminus I_0$. That is, $a(i)n(i)(a(i))^{-1} = e_i$ for all but finitely many $i \in I$, and so $ana^{-1} = \{a(i)n(i)(a(i))^{-1}\}_{i \in I} \in \prod_{i \in I}^{w} G_i$. Since $n \in N$ is arbitrary, $aNa^{-1} = a\left(\prod_{i \in I}^{w} G_i\right)a^{-1} \subseteq N = \prod_{i \in I}^{w} G_i$. Therefore, by Theorem I.5.1(iv), $N = \prod_{i \in I}^{w} G_i$ is a normal subgroup of $\prod_{i \in I} G_i$. □

Proposition I.8.5

Theorem I.8.5. Let $\{A_i \mid i \in I\}$ be a family of (additive) abelian groups. If B is an abelian group and $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism mapping the (external) direct sum $\sum A_i$ to B, $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i$ for all $i \in I$ and this property determines $\sum_{i\in I} A_i$ uniquely up to isomorphism. That is, $\sum_{i\in I} A_i$ is a coproduct in the category of abelian groups.

Proof. If $\{a_i\} \in \sum A_i$ is nonzero (that is, if $a_i \neq 0$ for some $i \in I$), then only finitely many of the a_i are nonzero (see Definition I.8.3), say $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$. Define $\psi : \sum A_i \to B$ as $\psi(0) = 0$ and

$$\psi(\{a_i\}) = \psi_{i_1}(a_{i_1}) + \psi_{i_2}(a_{i_2}) + \cdots + \psi_{i_r}(a_{i_r}) = \sum_{i\in I_0} \psi_i(a_i)$$

where $I_0 = \{i_1, i_2, \ldots, i_r\} = \{i \in I \mid a_i \neq 0\}$.
We leave as homework the verification that ψ is a homomorphism and $\psi\iota_i = \psi_i$ for all $i \in I$ (this is where commutivity is required).

Proposition I.8.5 (continued 1)

Proof (continued). For each $\{a_i\} \in \sum A_i$ with only finitely many nonzero a_i we have $\{a_i\} = \sum_{i\in I_0} \iota_i(a_i)$ where I_0 (a finite set, as required in the definition of external direct sum) is as above (since ι_i "embeds" each a_i into an $|I|$-tuple with only finitely many nonzero entries). Now for the uniqueness of ψ. If $\xi : \sum A_i \to B$ is a homomorphism such that $\xi\iota_i = \psi_i$ for all $i \in I$ then

$$\begin{aligned}
\xi(\{a_i\}) &= \xi\left(\sum_{i\in I_0} \iota_i(a_i)\right) && \text{by the observation above}\\
&= \sum_{i\in I_0} \xi\iota_i(a_i) && \text{since } \xi \text{ is a homomorphism}\\
&= \sum_{i\in I_0} \psi_i(a_i) && \text{by hypothesis of equality of } \xi\iota_i = \psi_i \text{ for all } i \in I\\
&= \sum_{i\in I_0} \psi\iota_i(a_i) && \text{by above ("homework")}
\end{aligned}$$

Proposition I.8.5 (continued 2)

Proof (continued).

$$\begin{aligned}
\xi(\{a_i\}) &= \sum_{i\in I_0} \psi\iota_i(a_i) && \text{by above}\\
&= \psi\left(\sum_{i\in I_0} \iota_i(a_i)\right) && \text{since } \psi \text{ is a homomorphism}\\
&= \psi(\{a_i\}) && \text{since } \{a_i\} = \sum_{i\in I_0} \iota_i(a_i).
\end{aligned}$$

So $\xi = \psi$ and ψ is unique. This uniqueness implies that $\sum A_i$ is a coproduct in the category of abelian groups (see Definition I.7.4).
By Theorem I.7.5, any two coproducts of $\{A_i\}_{i\in I}$ are equivalent. That is, there are morphisms (i.e., group homomorphisms) between the two coproducts which compose to give an identity mapping and hence the coproducts are isomorphic. □

Theorem I.8.6

Theorem I.8.6. Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of a group G such that

(i) $G = \langle \cup_{i\in I} N_i \rangle$;
(ii) for each $k \in I$, we have $N_k \cap \langle \cup_{i\neq k} N_i \rangle = \langle e \rangle$.

Then $G \cong \prod_{i\in I}^{w} N_i$.

Proof. If $\{a_i\} \in \prod^w N_i$ then (by Definition I.8.3) $a_i = e$ for all but a finite number of $i \in I$. Let I_0 be the finite set $\{i \in I \mid a_i \neq e\}$. Then $\prod_{i\in I_0} a_i$ is a well-defined element of G (i.e., independent of the order of the product) since for $a \in N_i$ and $b \in N_j$ (with $i \neq j$), $ab = ba$ by Theorem I.5.3(iv). Define $\varphi : \prod^w N_i \to G$ as $\varphi(\{a_i\}) = \prod_{i\in I_0} a_i$ (and $\varphi(\{e\}) = e$).

Theorem I.8.6 (continued 1)

Proof (continued). First, for $\{a_i\}, \{b_i\} \in \prod^w N_i$ where I_0 is as above and $I_1 = \{i \in I \mid b_i \neq e\}$, we have

$$\begin{aligned}
\varphi(\{a_i\}\{b_i\}) &= \varphi(\{a_i b_i\}) = \prod_{I_0 \cup I_1} (a_i b_i) \\
&= \left(\prod_{I_0 \cup I_1} a_i\right)\left(\prod_{I_0 \cup I_1} b_i\right) \text{ by the commutivity observation above} \\
&= \left(\prod_{I_0} a_i\right)\left(\prod_{I_1} b_i\right) \text{ since } a_i = e \text{ for } i \notin I_0, \text{ and } b_i = e \text{ for } i \notin I_1 \\
&= \varphi(\{a_i\})\varphi(\{b_i\})
\end{aligned}$$

so φ is a homomorphism.

Theorem I.8.6 (continued 2)

Proof (continued). For $a_k \in N_k$ we have

$$\begin{aligned}
\varphi\iota_k(a_k) &= \varphi(\{a_i\}_{i\in I}) \text{ where } a_i = e \text{ for } i \neq k \\
&= \prod a_k \text{ since } a_k \text{ is the only non-}e\text{ element} \\
&= a_k.
\end{aligned}$$

That is (in terms of i), $\varphi\iota_i(a_i) = a_i$ for all $a_i \in N_i$.
Next, to show φ is onto. Since G is generated by the subgroups N_i, every element a of G is a finite product of elements from various N_i (see Theorem I.2.8). Since elements of N_i and N_j commute (for $i \neq j$) as mentioned above, a can be written as a product

$$a = \prod_{i \in I_0} a_i \text{ where } a_i \in N_i \qquad (*)$$

and I_0 is some finite subset of I.

Theorem I.8.6 (continued 3)

Proof (continued). Hence $\prod_{i \in I_0} \iota_i(a_i) \in \prod^w_{i\in I} N_i$ and

$$\begin{aligned}
\varphi\left(\prod_{i\in I_0} \iota_i(a_i)\right) &= \prod_{i\in I_0} \varphi\iota_i(a_i) \text{ since } \varphi \text{ is a homomorphism} \\
&= \prod_{i \in I_0} a_i \text{ since } \varphi\iota_i(a_i) = a_i \text{ as shown above} \\
&= a \text{ by } (*).
\end{aligned}$$

So φ is onto.
Now to show that φ is one to one. Suppose $\varphi(\{a_i\}) = \prod_{i\in I_0} a_i = e \in G$. WLOG we take $I_0 = \{1, 2, \ldots, n\}$. Then $\prod_{i \in I_0} a_i = a_1 a_2 \cdots a_n = e$ with $a_i \in N_i$. Hence $a_1^{-1} = a_2 a_3 \cdots a_n \in N_1 \cap \langle \cup_{i=2}^n N_i \rangle = \langle e \rangle$ and therefore $a_1^{-1} = e$ and $a_1 = e$. Similarly $a_i = e$ for all $i \in I_0$ (and also for all $i \in I$).
So $\mathrm{Ker}(\varphi) = \{e\}$ and by Theorem I.2.3(i) φ is one to one.
Hence, φ is an isomorphism and $G \cong \prod^w_{i \in I} N_i$. □

Corollary I.8.11

Corollary I.8.11. Let $\{G_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be families of groups such that N_i is a normal subgroup of G_i for each $i \in I$.

(i) $\prod N_i$ is a normal subgroup of $\prod G_i$ and $\prod G_i / \prod N_i \cong \prod G_i/N_i$.
(ii) $\prod^w N_i$ is a normal subgroup of $\prod^w G_i$ and $\prod^w G_i / \prod^w N_i \cong \prod^w G_i/N_i$.

Proof of (i). For each $i \in I$, let $\pi_i : G_i \to G_i/N_i$ be the canonical epimorphism (see Theorem I.5.5). By Theorem I.8.10, the map $\prod \pi_i : \prod G_i \to \prod G_i/N_i$ is an epimorphism with kernel $\prod N_i$. Then $\prod G_i / \prod N_i \cong \prod G_i/N_i$ by the First Isomorphism Theorem (Corollary I.5.7) since $\mathrm{Im}(\prod \pi_i) = \prod G_i/N_i$ because $\prod \pi_i$ is onto. □