Modern Algebra

Chapter I. Groups

I.9. Free Groups, Free Products, Generators and Relations—Proofs of Theorems

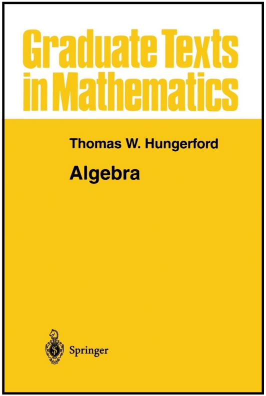

Table of contents

Theorem I.9.1

Theorem I.9.1. If X is a nonempty set and $F = F(X)$ is the set of all reduced words on X, then F is a group under the binary operation defined in the previous definition. Also, $F = \langle X \rangle$ (where $\langle X \rangle$ represents the group generated by set X). The group $F = F(X)$ is called the *free group on set* X.

Proof. 1 is the identity and word $x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}$ has inverse $x_n^{-\delta_n} x_{n-1}^{-\delta_{n-1}} \cdots x_1^{-\delta_1}$. We now show associativity.

Theorem I.9.1

Theorem I.9.1. If X is a nonempty set and $F = F(X)$ is the set of all reduced words on X, then F is a group under the binary operation defined in the previous definition. Also, $F = \langle X \rangle$ (where $\langle X \rangle$ represents the group generated by set X). The group $F = F(X)$ is called the *free group on set* X.

Proof. 1 is the identity and word $x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}$ has inverse $x_n^{-\delta_n} x_{n-1}^{-\delta_{n-1}} \cdots x_1^{-\delta_1}$. We now show associativity.
For $x \in X$ and $\delta = \pm 1$, denote by $|x^\delta|$ the map from F to F given by the mappings $1 \mapsto x^\delta$ and

$$x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n} \mapsto \begin{cases} x^\delta x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n} & \text{if } x^\delta \neq x_1^{-\delta_1} \\ x_2^{\delta_2} x_3^{\delta_3} \cdots x_n^{\delta_n} & \text{if } x^\delta = x_1^{-\delta_1}, n > 1 \\ 1 & \text{if } x^\delta = x_1^{-\delta_1} \text{ and } n = 1. \end{cases}$$

Theorem I.9.1

Theorem I.9.1. If X is a nonempty set and $F = F(X)$ is the set of all reduced words on X, then F is a group under the binary operation defined in the previous definition. Also, $F = \langle X \rangle$ (where $\langle X \rangle$ represents the group generated by set X). The group $F = F(X)$ is called the *free group on set* X.

Proof. 1 is the identity and word $x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}$ has inverse $x_n^{-\delta_n} x_{n-1}^{-\delta_{n-1}} \cdots x_1^{-\delta_1}$. We now show associativity.
For $x \in X$ and $\delta = \pm 1$, denote by $|x^\delta|$ the map from F to F given by the mappings $1 \mapsto x^\delta$ and

$$x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n} \mapsto \begin{cases} x^\delta x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n} & \text{if } x^\delta \neq x_1^{-\delta_1} \\ x_2^{\delta_2} x_3^{\delta_3} \cdots x_n^{\delta_n} & \text{if } x^\delta = x_1^{-\delta_1}, n > 1 \\ 1 & \text{if } x^\delta = x_1^{-\delta_1} \text{ and } n = 1. \end{cases}$$

Theorem I.9.1 (continued 1)

Proof (continued). So mapping $|x^\delta|$ just multiplies a word on the left by x^δ and then reduces the word. So mapping $|x||x^{-1}| = 1_F = |x^{-1}||x|$ and more generally $|x^\delta||x^{-\delta}| = 1_F = |x^{-\delta}||x^\delta|$. Hence each mapping $|x^\delta|$ has a two-sided inverse. By Corollary 0.3.B, $|x^\delta|$ is then a bijection on F, and hence is a permutation of F. Let $A(F)$ be the group of all permutations of F and let F_0 be the subgroup of $A(F)$ generated by the set $\{|x| \mid x \in X\}$. Consider the map $\varphi : F \to F_0$ given by $\varphi(1) = 1_F$ and $\varphi(x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}) = |x_1^{\delta_1}||x_2^{\delta_2}| \cdots |x_n^{\delta_n}|$ (so φ maps reduced words to permutations of F).

Theorem I.9.1 (continued 1)

Proof (continued). So mapping $|x^\delta|$ just multiplies a word on the left by x^δ and then reduces the word. So mapping $|x||x^{-1}| = 1_F = |x^{-1}||x|$ and more generally $|x^\delta||x^{-\delta}| = 1_F = |x^{-\delta}||x^\delta|$. Hence each mapping $|x^\delta|$ has a two-sided inverse. By Corollary 0.3.B, $|x^\delta|$ is then a bijection on F, and hence is a permutation of F. Let $A(F)$ be the group of all permutations of F and let F_0 be the subgroup of $A(F)$ generated by the set $\{|x| \mid x \in X\}$. Consider the map $\varphi : F \to F_0$ given by $\varphi(1) = 1_F$ and $\varphi(x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}) = |x_1^{\delta_1}||x_2^{\delta_2}| \cdots |x_n^{\delta_n}|$ (so φ maps reduced words to permutations of F). Since F_0 is generated by $\{|x| \mid x \in X\}$, then the elements of F_0 are of the form $|x_1^{\delta_1}||x_2^{\delta_2}| \cdots |x_n^{\delta_n}|$ for some $x_i \in X$ and $\delta_i = \pm 1$ by Theorem I.2.8. So "clearly" φ is onto F_0 (i.e., φ is a surjection).

Theorem I.9.1 (continued 1)

Proof (continued). So mapping $|x^\delta|$ just multiplies a word on the left by x^δ and then reduces the word. So mapping $|x||x^{-1}| = 1_F = |x^{-1}||x|$ and more generally $|x^\delta||x^{-\delta}| = 1_F = |x^{-\delta}||x^\delta|$. Hence each mapping $|x^\delta|$ has a two-sided inverse. By Corollary 0.3.B, $|x^\delta|$ is then a bijection on F, and hence is a permutation of F. Let $A(F)$ be the group of all permutations of F and let F_0 be the subgroup of $A(F)$ generated by the set $\{|x| \mid x \in X\}$. Consider the map $\varphi : F \to F_0$ given by $\varphi(1) = 1_F$ and $\varphi(x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}) = |x_1^{\delta_1}||x_2^{\delta_2}| \cdots |x_n^{\delta_n}|$ (so φ maps reduced words to permutations of F). Since F_0 is generated by $\{|x| \mid x \in X\}$, then the elements of F_0 are of the form $|x_1^{\delta_1}||x_2^{\delta_2}| \cdots |x_n^{\delta_n}|$ for some $x_i \in X$ and $\delta_i = \pm 1$ by Theorem I.2.8. So "clearly" φ is onto F_0 (i.e., φ is a surjection).

Theorem I.9.1 (continued 2)

Proof (continued). Next, let $w_1, w_2 \in F$. Then we claim $\varphi(w_1 w_2) = \varphi(w_1)\varphi(w_2)$. To see this, notice that $w_1 w_2$ is reduced and so some terms on the right-hand end of w_1 may have been cancelled with the same number of terms on the left-hand end of w_2 (there are k such terms, where k is as given in the definition of reduced word product above). Say $w_1 = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ and $w_2 = y_1^{\delta_1} y_2^{\delta_2} \cdots y_m^{\delta_m}$. In the product $w_1 w_2$, say the cancelled terms are

$$x_k^{\lambda_k} x_{k+1}^{\lambda_{k+1}} \cdots x_n^{\lambda_n} y_1^{\delta_1} y_2^{\delta_2} \cdots y_{n-k+1}^{\delta_{n-k+1}} = 1.$$

Notice $\varphi(1) = 1_F$ so that

$$|x_k^{\lambda_k}||x_{k+1}^{\lambda_{k+1}}| \cdots |x_n^{\lambda_n}||y_1^{\delta_1}||y_2^{\delta_2}| \cdots |y_{n-k+1}^{\delta_{n-k+1}}| = 1_F.$$

Also

$$\begin{aligned} w_1 w_2 &= x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} y_1^{\delta_1} y_2^{\delta_2} \cdots y_m^{\delta_m} \text{ before reducing} \\ &= x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{k-1}^{\lambda_{k-1}} y_{n-k+2}^{\delta_{n-k+2}} \cdots y_m^{\delta_m} \text{ after reducing.} \end{aligned}$$

Theorem I.9.1 (continued 2)

Proof (continued). Next, let $w_1, w_2 \in F$. Then we claim $\varphi(w_1 w_2) = \varphi(w_1)\varphi(w_2)$. To see this, notice that $w_1 w_2$ is reduced and so some terms on the right-hand end of w_1 may have been cancelled with the same number of terms on the left-hand end of w_2 (there are k such terms, where k is as given in the definition of reduced word product above). Say $w_1 = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ and $w_2 = y_1^{\delta_1} y_2^{\delta_2} \cdots y_m^{\delta_m}$. In the product $w_1 w_2$, say the cancelled terms are

$$x_k^{\lambda_k} x_{k+1}^{\lambda_{k+1}} \cdots x_n^{\lambda_n} y_1^{\delta_1} y_2^{\delta_2} \cdots y_{n-k+1}^{\delta_{n-k+1}} = 1.$$

Notice $\varphi(1) = 1_F$ so that

$$|x_k^{\lambda_k}||x_{k+1}^{\lambda_{k+1}}| \cdots |x_n^{\lambda_n}||y_1^{\delta_1}||y_2^{\delta_2}| \cdots |y_{n-k+1}^{\delta_{n-k+1}}| = 1_F.$$

Also

$$\begin{aligned} w_1 w_2 &= x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} y_1^{\delta_1} y_2^{\delta_2} \cdots y_m^{\delta_m} \text{ before reducing} \\ &= x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{k-1}^{\lambda_{k-1}} y_{n-k+2}^{\delta_{n-k+2}} \cdots y_m^{\delta_m} \text{ after reducing.} \end{aligned}$$

Theorem I.9.1 (continued 3)

Proof (continued). So

$$\begin{aligned}\varphi(w_1 w_2) &= |x_1^{\lambda_1}||x_2^{\lambda_2}|\cdots|x_{k-1}^{\lambda_{k-1}}||y_{n-k+2}^{\delta_{n-k+2}}|\cdots|y_m^{\delta_m}| \\ &= |x_1^{\lambda_1}||x_2^{\lambda_2}|\cdots|x_{k-1}^{\lambda_{k-1}}|(|x_k^{\lambda_k}|\cdots|x_n^{\lambda_n}||y_1^{\delta_1}|\cdots|y_{n-k+1}^{\delta_{n-k+1}}|) \\ &\quad |y_{n-k+2}^{\delta_{n-k+2}}|\cdots|y_m^{\delta_m}| \\ &\quad \text{since } (|x_k^{\lambda_k}|\cdots|x_n^{\lambda_n}||y_1^{\delta_1}|\cdots|y_{n-k+1}^{\delta_{n-k+1}}) = 1_F \\ &= (|x_1^{\lambda_1}||x_2^{\lambda_2}|\cdots|x_n^{\lambda_n}|)(|y_1^{\delta_1}||y_2^{\delta_2}|\cdots|y_m^{\delta_m}|) \text{ since function} \\ &\quad \text{composition is associative} \\ &= \varphi(w_1)\varphi(w_2).\end{aligned}$$

Hence φ has the homomorphism property.

Theorem I.9.1 (continued 4)

Theorem I.9.1. If X is a nonempty set and $F = F(X)$ is the set of all reduced words on X, then F is a group under the binary operation defined in the previous definition. Also, $F = \langle X \rangle$ (where $\langle X \rangle$ represents the group generated by set X). The group $F = F(X)$ is called the *free group on set* X.

Proof (continued). Now suppose $\varphi(x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}) = 1_F$. Then $|x_1^{\delta_1}||x_2^{\delta_2}| \cdots |x_n^{\delta_n}| = 1_F$. Since, in group F_0, the inverse of $|x^{\delta}|$ is $|x^{-\delta}|$, it follows that $x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n} = 1$ in F. So $\mathrm{Ker}(\varphi) = 1$ and by Theorem I.2.3(i), φ is one to one (injective). Since F_0 is a group, then the binary operation in F_0 is associative. Since φ is a one to one and onto mapping with the homomorphism property, then φ is actually a group isomorphism and so F is a group and hence associativity holds in F.

Of course, F is generated by X since the "alphabet" of group F (the "letters" which make up the words) is determined by X as $X \cup X^{-1} \cup \{1\}$. □

Theorem I.9.1 (continued 4)

Theorem I.9.1. If X is a nonempty set and $F = F(X)$ is the set of all reduced words on X, then F is a group under the binary operation defined in the previous definition. Also, $F = \langle X \rangle$ (where $\langle X \rangle$ represents the group generated by set X). The group $F = F(X)$ is called the *free group on set* X.

Proof (continued). Now suppose $\varphi(x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}) = 1_F$. Then $|x_1^{\delta_1}||x_2^{\delta_2}| \cdots |x_n^{\delta_n}| = 1_F$. Since, in group F_0, the inverse of $|x^\delta|$ is $|x^{-\delta}|$, it follows that $x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n} = 1$ in F. So $\mathrm{Ker}(\varphi) = 1$ and by Theorem I.2.3(i), φ is one to one (injective). Since F_0 is a group, then the binary operation in F_0 is associative. Since φ is a one to one and onto mapping with the homomorphism property, then φ is actually a group isomorphism and so F is a group and hence associativity holds in F.

Of course, F is generated by X since the "alphabet" of group F (the "letters" which make up the words) is determined by X as $X \cup X^{-1} \cup \{1\}$. □

Theorem I.9.2

Theorem I.9.2. Let F be the free group on set X and $\iota : X \to F$ the inclusion map (see page 4). If G is a group and $f : X \to G$ is a map of sets, then there exists a unique homomorphism of groups $\overline{f} : F \to G$ such that $\overline{f}\iota = f$. In other words, F is a free object on the set X is the category of groups.

Proof. Define $\overline{f}(1) = e$ and for $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ a nonempty reduced word on X define $\overline{f}(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}) = f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_n)^{\lambda_n}$. Now f is a set function but G is a group and $\lambda_i = \pm 1$, so $f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_n)^{\lambda_n}$ is well-defined.

Theorem I.9.2

Theorem I.9.2. Let F be the free group on set X and $\iota : X \to F$ the inclusion map (see page 4). If G is a group and $f : X \to G$ is a map of sets, then there exists a unique homomorphism of groups $\overline{f} : F \to G$ such that $\overline{f}\iota = f$. In other words, F is a free object on the set X is the category of groups.

Proof. Define $\overline{f}(1) = e$ and for $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ a nonempty reduced word on X define $\overline{f}(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}) = f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_n)^{\lambda_n}$. Now f is a set function but G is a group and $\lambda_i = \pm 1$, so $f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_n)^{\lambda_n}$ is well-defined. Now to show that $\overline{f}$ satisfies the homomorphism property. Let $w_1 = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ and $w_2 = y_1^{\delta_1} y_2^{\delta_2} \cdots y_m^{\delta_m}$ be reduced words and, as in the proof of Theorem I.9.1, let

$$x_i^{\lambda_i} x_{i+1}^{\lambda_{i+1}} \cdots x_n^{\lambda_n} y_1^{\delta_1} y_2^{\delta_2} \cdots y_{n-i+1}^{\delta_{n-i+1}}$$

be the cancelled terms in the product of these two words.

Theorem I.9.2

Theorem I.9.2. Let F be the free group on set X and $\iota : X \to F$ the inclusion map (see page 4). If G is a group and $f : X \to G$ is a map of sets, then there exists a unique homomorphism of groups $\overline{f} : F \to G$ such that $\overline{f}\iota = f$. In other words, F is a free object on the set X is the category of groups.

Proof. Define $\overline{f}(1) = e$ and for $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ a nonempty reduced word on X define $\overline{f}(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}) = f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_n)^{\lambda_n}$. Now f is a set function but G is a group and $\lambda_i = \pm 1$, so $f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_n)^{\lambda_n}$ is well-defined. Now to show that $\overline{f}$ satisfies the homomorphism property. Let $w_1 = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ and $w_2 = y_1^{\delta_1} y_2^{\delta_2} \cdots y_m^{\delta_m}$ be reduced words and, as in the proof of Theorem I.9.1, let

$$x_i^{\lambda_i} x_{i+1}^{\lambda_{i+1}} \cdots x_n^{\lambda_n} y_1^{\delta_1} y_2^{\delta_2} \cdots y_{n-i+1}^{\delta_{n-i+1}}$$

be the cancelled terms in the product of these two words.

Theorem I.9.2 (continued 1)

Proof (continued) Then the product of these cancelled terms equals the word 1 and so in group G

$$\overline{f}(1) = f(x_i)^{\lambda_i} f(x_{i+1})^{\lambda_{i+1}} \cdots f(x_n)^{\lambda_n} f(y_1)^{\delta_1} f(y_2)^{\delta_2} \cdots f(y_{n-i+1})^{\delta_{n-i+1}} = e.$$

So

$$\begin{aligned}
\overline{f}(w_1 w_2) &= \overline{f}(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{i-1}^{\lambda_{i-1}} y_{n-i+2}^{\delta_{n-i+2}} y_{n-i+3}^{\delta_{n-i+3}} \cdots y_m^{\delta_m}) \\
&= f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_{i-1})^{\lambda_{i-1}} \\
&\quad f(y_{n-i+2})^{\delta_{n-i+2}} f(y_{n-i+3})^{\delta_{n-i+3}} \cdots f(y_m)^{\delta_m} \\
&= f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_{i-1})^{\lambda_{i-1}} e \\
&\quad f(y_{n-i+2})^{\delta_{n-i+2}} f(y_{n-i+3})^{\delta_{n-i+3}} \cdots f(y_m)^{\delta_m} \\
&= f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_{i-1})^{\lambda_{i-1}} \left(f(x_i)^{\lambda_i} \cdots f(x_n)^{\lambda_n} \right. \\
&\quad \left. f(y_1)^{\delta_1} f(y_2)^{\delta_2} \cdots f(y_{n-1+i})^{\delta_{n-i+1}} \right) f(y_{n-i+2})^{\delta_{n-i+2}} \cdots f(y_m)^{\delta_m}
\end{aligned}$$

Theorem I.9.2 (continued 1)

Proof (continued) Then the product of these cancelled terms equals the word 1 and so in group G

$$\overline{f}(1) = f(x_i)^{\lambda_i} f(x_{i+1})^{\lambda_{i+1}} \cdots f(x_n)^{\lambda_n} f(y_1)^{\delta_1} f(y_2)^{\delta_2} \cdots f(y_{n-i+1})^{\delta_{n-i+1}} = e.$$

So

$$\begin{aligned}
\overline{f}(w_1 w_2) &= \overline{f}(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{i-1}^{\lambda_{i-1}} y_{n-i+2}^{\delta_{n-i+2}} y_{n-i+3}^{\delta_{n-i+3}} \cdots y_m^{\delta_m}) \\
&= f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_{i-1})^{\lambda_{i-1}} \\
&\quad f(y_{n-i+2})^{\delta_{n-i+2}} f(y_{n-i+3})^{\delta_{n-i+3}} \cdots f(y_m)^{\delta_m} \\
&= f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_{i-1})^{\lambda_{i-1}} e \\
&\quad f(y_{n-i+2})^{\delta_{n-i+2}} f(y_{n-i+3})^{\delta_{n-i+3}} \cdots f(y_m)^{\delta_m} \\
&= f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_{i-1})^{\lambda_{i-1}} \Big(f(x_i)^{\lambda_i} \cdots f(x_n)^{\lambda_n} \\
&\quad f(y_1)^{\delta_1} f(y_2)^{\delta_2} \cdots f(y_{n-1+i})^{\delta_{n-i+1}} \Big) f(y_{n-i+2})^{\delta_{n-i+2}} \cdots f(y_m)^{\delta_m}
\end{aligned}$$

Theorem I.9.2 (continued 2)

Proof (continued)

$$\begin{aligned}\overline{f}(w_1w_2) &= f(x_1)^{\lambda_1}f(x_2)^{\lambda_2}\cdots f(x_{i-1})^{\lambda_{i-1}}f(x_i)^{\lambda_i}\cdots f(x_n)^{\lambda_n}\\ &\quad f(y_1)^{\delta_1}f(y_2)^{\delta_2}\cdots f(y_{n-1+i})^{\delta_{n-i+1}}f(y_{n-i+2})^{\delta_{n-i+2}}\cdots f(y_m)^{\delta_m}\\ &= \overline{f}(w_1)\overline{f}(w_2)\end{aligned}$$

and so $\overline{f}$ has the homomorphism property.

Now the inclusion map ι simply "embeds" the set of "letters" X into the set of reduced words F. $\overline{f}$ maps the reduced words to group G, so $\overline{f}\iota$ maps X to G and has the homomorphism property. Also, by definition, $\overline{f}\iota = f$ on set X.

Theorem I.9.2 (continued 2)

Proof (continued)

$$\begin{aligned}\overline{f}(w_1w_2) &= f(x_1)^{\lambda_1}f(x_2)^{\lambda_2}\cdots f(x_{i-1})^{\lambda_{i-1}}f(x_i)^{\lambda_i}\cdots f(x_n)^{\lambda_n}\\ &\quad f(y_1)^{\delta_1}f(y_2)^{\delta_2}\cdots f(y_{n-1+i})^{\delta_{n-i+1}}f(y_{n-i+2})^{\delta_{n-i+2}}\cdots f(y_m)^{\delta_m}\\ &= \overline{f}(w_1)\overline{f}(w_2)\end{aligned}$$

and so $\overline{f}$ has the homomorphism property.

Now the inclusion map ι simply "embeds" the set of "letters" X into the set of reduced words F. $\overline{f}$ maps the reduced words to group G, so $\overline{f}\iota$ maps X to G and has the homomorphism property. Also, by definition, $\overline{f}\iota = f$ on set X.

Theorem I.9.2 (continued 3)

Theorem I.9.2. Let F be the free group on set X and $\iota : X \to F$ the inclusion map (see page 4). If G is a group and $f : X \to G$ is a map of sets, then there exists a unique homomorphism of groups $\overline{f} : F \to G$ such that $\overline{f}\iota = f$. In other words, F is a free object on the set X is the category of groups.

Proof (continued) If g is any homomorphism mapping set of reduced words F to group G such that $g\iota = f$ on set X, then

$$\begin{aligned} g(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}) &= g(x_1)^{\lambda_1} g(x_2)^{\lambda_2} \cdots g(x_n)^{\lambda_n} \text{ since } g \text{ is a homomorphism} \\ &= g\iota(x_1)^{\lambda_1} g\iota(x_2)^{\lambda_2} \cdots g\iota(x_n)^{\lambda_n} \text{ since } g\iota = g \text{ on } X \\ &= f(x_1)^{\lambda_1} f(x_2)^{\lambda_2} \cdots f(x_n)^{\lambda_n} \text{ by assumption} \\ &= \overline{f}(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}) \text{ by the definition of } \overline{f}. \end{aligned}$$

So the homomorphism $\overline{f} = F \to G$ is unique. □

Corollary I.9.3

Corollary I.9.3. Every group G is the homomorphic image of a free group.

Proof. Let X be a set of generators of G (such a set exists since G itself generates G). Let F be the free group on the set X. The inclusion map $\iota : X \to G$ is such that $x \in X$ is mapped to $x \in G$. By Theorem I.9.2, there is homomorphism $\overline{f} : F \to G$ which maps $x \in X$ as $x \mapsto x \in G$. Since $G = \langle X \rangle$ (by the choice of set X) then $\overline{f}$ is onto G (i.e., $\overline{f}$ is an epimorphism). So $\overline{f}$ maps free group F onto group G (that is, $\mathrm{Im}(\overline{f}) = G$) and $\overline{f}$ is a homomorphism. □

Corollary I.9.3

Corollary I.9.3. Every group G is the homomorphic image of a free group.

Proof. Let X be a set of generators of G (such a set exists since G itself generates G). Let F be the free group on the set X. The inclusion map $\iota : X \to G$ is such that $x \in X$ is mapped to $x \in G$. By Theorem I.9.2, there is homomorphism $\overline{f} : F \to G$ which maps $x \in X$ as $x \mapsto x \in G$. Since $G = \langle X \rangle$ (by the choice of set X) then $\overline{f}$ is onto G (i.e., $\overline{f}$ is an epimorphism). So $\overline{f}$ maps free group F onto group G (that is, $\mathrm{Im}(\overline{f}) = G$) and $\overline{f}$ is a homomorphism. □

Theorem I.9.5

Theorem I.9.5. von Dyck's Theorem.
Let X be a set, Y a set of reduced words on X and G the group defined by the generators $x \in X$ and relations $w = e$ for $w \in Y$. If H is any group such that $H = \langle X \rangle$ and H satisfies all the relations $w = e$ for $w \in Y$, then there is an epimorphism mapping $G \to H$.

Proof. If F is the free group on X then, as in the proof of Corollary I.9.3, the inclusion map $\iota : X \to H$ induces an epimorphism $\varphi : F \to H$.

Theorem I.9.5

Theorem I.9.5. von Dyck's Theorem.
Let X be a set, Y a set of reduced words on X and G the group defined by the generators $x \in X$ and relations $w = e$ for $w \in Y$. If H is any group such that $H = \langle X \rangle$ and H satisfies all the relations $w = e$ for $w \in Y$, then there is an epimorphism mapping $G \to H$.

Proof. If F is the free group on X then, as in the proof of Corollary I.9.3, the inclusion map $\iota : X \to H$ induces an epimorphism $\varphi : F \to H$. Since the relations $w = e$ hold for all $w \in Y$, then by Note I.9.3 $w \in \mathrm{Ker}(\varphi)$ and so $Y \subseteq \mathrm{Ker}(\varphi)$ (here the elements of Y are interpreted both as words on X and products in group H, as Hungerford remarks on page 67). So the normal subgroup N generated by Y in F is contained in $\mathrm{Ker}(\varphi)$; that is, $N \subseteq \mathrm{Ker}(\varphi)$ or equivalently $\varphi(N) < \{0\}$ where 0 is the identity of group H. Now $\varphi : F \to H$ is an epimorphism, $N \lhd F$, $\{0\} \lhd H$, and $\varphi(N) < \{0\}$. So by Corollary I.5.8, φ induces a homomorphism ψ mapping $F/N \to H/\{0\} \cong H$.

Theorem I.9.5

Theorem I.9.5. von Dyck's Theorem.
Let X be a set, Y a set of reduced words on X and G the group defined by the generators $x \in X$ and relations $w = e$ for $w \in Y$. If H is any group such that $H = \langle X \rangle$ and H satisfies all the relations $w = e$ for $w \in Y$, then there is an epimorphism mapping $G \to H$.

Proof. If F is the free group on X then, as in the proof of Corollary I.9.3, the inclusion map $\iota : X \to H$ induces an epimorphism $\varphi : F \to H$. Since the relations $w = e$ hold for all $w \in Y$, then by Note I.9.3 $w \in \text{Ker}(\varphi)$ and so $Y \subseteq \text{Ker}(\varphi)$ (here the elements of Y are interpreted both as words on X and products in group H, as Hungerford remarks on page 67). So the normal subgroup N generated by Y in F is contained in $\text{Ker}(\varphi)$; that is, $N \subseteq \text{Ker}(\varphi)$ or equivalently $\varphi(N) < \{0\}$ where 0 is the identity of group H. Now $\varphi : F \to H$ is an epimorphism, $N \triangleleft F$, $\{0\} \triangleleft H$, and $\varphi(N) < \{0\}$. So by Corollary I.5.8, φ induces a homomorphism ψ mapping $F/N \to H/\{0\} \cong H$.

Theorem I.9.5 (continued)

Theorem I.9.5. von Dyck's Theorem.
Let X be a set, Y a set of reduced words on X and G the group defined by the generators $x \in X$ and relations $w = e$ for $w \in Y$. If H is any group such that $H = \langle X \rangle$ and H satisfies all the relations $w = e$ for $w \in Y$, then there is an epimorphism mapping $G \to H$.

Proof (continued). Now for $aN \in F/N$ we have (again by Corollary I.5.8) that $\psi(aN) = \varphi(a)\{0\}$. Since φ is onto group H then ψ is onto group $H/\{0\} \cong H$. That is, $\psi : G \to H$ is an epimorphism. (Recall $G \cong F/N$. Technically, ψ has to be composed with an isomorphism mapping $G \to F/N$ and an isomorphism mapping $H/\{0\} \to H$.) □