Modern Algebra

Chapter II. The Structure of Groups

II.3. The Krull-Schmidt Theorem—Proofs of Theorems

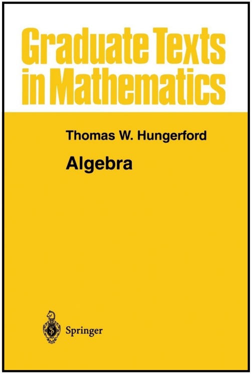

Table of contents

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$$S = \{H \triangleleft G \mid G \cong H \times T_H \text{ for some } T_H < G, \text{ and } H \text{ is not isomorphic to a finite direct product of indecomposable subgroups}\}.$$

Then $G \in S$ so $S \neq \varnothing$.

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Then $G \in S$ so $S \neq \varnothing$. If $H \in S$ then H is not a finite direct product of indecomposable subgroups (in particular, H is not a "product" of one indecomposable group), so H is not indecomposable. That is, H can be "decomposed"; i.e., there exists proper subgroups K_H and J_H of H such that $H \cong K_H \times J_H$.

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Theorem II.3.3 (continued 1)

Proof (continued). So H is a direct factor of G, and K_H and J_H are direct factors of H, so by Exercise I.8.12(a), K_H and J_H are normal in G. Since H is not isomorphic to a finite direct product of indecomposable subgroups, then either K_H or J_H (without loss of generality, say K_H) must not be isomorphic to a finite direct product of indecomposable subgroups. Since $G \cong H \times T_H \cong K_H \times J_H \times T_H$, then G has a subgroup J_{G_0} isomorphic to $J_H \times T_H$ such that $G \cong K_H \times J_{G_0}$ and so $K_H \in S$ (notice that, by Exercise I.8.12(a), K_H and J_{G_0} are normal subgroups of G). That is, for each $H \in S$ there is a proper subset K_H of H in S.

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Proof (continued). So we have a CONTRADICTION in the case that G satisfies the descending chain condition. To complete the proof, we still need a contradiction in the case that G satisfies the ascending chain condition. We now have by induction that for each $n \in \mathbb{N}$, $G \cong G_n \times J_{G_{n-1}} \times J_{G_{n-2}} \times \cdots \times J_{G_0}$ where each J_{G_i} is a proper subgroup of G (notice that $J_{G_0} \cong J_H \times T_H$ in the notation above and that $H \cong G_n \times J_{G_n} \times \cdots \times J_{G_1}$).

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Lemma II.3.4. Let G be a group that satisfies the ascending chain condition on normal subgroups and let f be an endomorphism of G. Then f is an automorphism if and only if f is an epimorphism (i.e., an onto homomorphism). Let G be a group that satisfies the descending chain condition on normal subgroups and let f be a normal endomorphism of G. Then f is an automorphism if and only if f is a monomorphism (i.e., a one to one homomorphism).

Proof. Suppose G satisfies the ascending chain condition and that f is an epimorphism. Since a composition of onto functions is onto, then $f^k = ff \cdots f$ is also an epimorphism of G. Recall that if $f : G \longrightarrow H$ is a homomorphism, then $\text{Ker}(f)$ is a subgroup of H by Exercise I.2.9(a) and $\text{Ker}(f) \lhd G$ by Theorem I.5.5.

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Proof (continued). If $a \in G$ and $f(a) = e$, then $a = f^n(b)$ for some $b \in G$ since f^n is onto, and $e = f(a) = f(f^n(b)) = f^{n+1}(b)$. Consequently, $b \in \text{Ker}(f^{n+1}) = \text{Ker}(f^n)$ which implies that $a = f^n(b) = e$; that is, $f(a) = e$ implies $a = e$. So $\text{Ker}(f) = \{e\}$ and by Theorem I.2.3(i), f is a monomorphism (one to one homomorphism). So $f : G \to G$ is a one to one and onto homomorphism and hence is an automorphism of G.

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Suppose G satisfies the descending chain condition and that f is a monomorphism. Since f is a normal endomorphism, then for $k \geq 1$ and for all $a \in G$ we have for any $b \in G$ that

$$af^k(b)a^{-1} = af(f^{k-1}(b))1^{-1} = f(af^{k-1}(b)a^{-1})$$
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So $\text{Im}(f^k)a^{-1} \subset \text{Im}(f^k)$ for all $a \in G$, and by Theorem I.5.1(iv), $\text{Im}(f^k) \lhd G$. So we have the descending chain $G > \text{Im}(f) > \text{Im}(f^2) > \cdots$ and by hypothesis, $G_i = G_n$ for some $n \in \mathbb{N}$ and for all $i \geq n$, which implies that $\text{Im}(f^n) = \text{Im}(f^{n+1})$.

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Proof (continued). Then for any $a \in G$, we have $f^n(a) = f^{n+1}(b)$ for some $b \in G$ (since the images of f^n and f^{n+1} are the same). A composition of one to one maps is one to one, so that the fact that f is a monomomomorphism implies that f^n is also a monomorphism, so $f^n(a) = f^{n+1}(b) = f^n(f(b))$ implies $a = f(b)$. That is, any $a \in G$ is the image under f of some $b \in G$ and so f is onto. Therefore, $f : G \to G$ is a one to one and onto homomorphism and hence is an automorphism of G. □

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Lemma II.3.5

Lemma II.3.5. If G is a group that satisfies both the ascending and descending chain conditions on normal subgroups an df is a normal endomorphism of G, then for some $n \geq 1$, we have $G = \text{Ker}(f^n) \times \text{Im}(f^n)$.

Proof. As shown in the second part of the proof of Lemma II.3.4, since f is a normal endomorphism, then for all $k \geq 1$ we have $\text{Im}(f^k) \triangleleft G$. Also as in the proof of Lemma II.3.4, we have two chains of normal subgroups, $G > \text{Im}(f) > \text{Im} f^2 > \cdots$ and $\langle e \rangle < \text{Ker}(f) < \text{Ker}(f^2) < \cdots$.

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Suppose $a \in \text{Ker}(f^n) \cap \text{Im}(f^n)$. Then $a = f^n(b)$ for some $b \in G$ (since $a \in \text{Im}(f^n)$) and so $f^{2n}(b) = f^n(f^n(b)) = f^n(a) = e$ (since $a \in \text{Ker}(f^n)$). Consequently, $b \in \text{Ker}(f^{2n}) = \text{Ker}(f^n)$ and so $a = f^n(b) = e$. Therefore $\text{Ker}(f^n) \cap \text{Im}(f^n) = \langle e \rangle$.

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Lemma II.3.5. If G is a group that satisfies both the ascending and descending chain conditions on normal subgroups an df is a normal endomorphism of G, then for some $n \geq 1$, we have $G = \mathrm{Ker}(f^n) \times \mathrm{Im}(f^n)$.

Proof. As shown in the second part of the proof of Lemma II.3.4, since f is a normal endomorphism, then for all $k \geq 1$ we have $\mathrm{Im}(f^k) \triangleleft G$. Also as in the proof of Lemma II.3.4, we have two chains of normal subgroups, $G > \mathrm{Im}(f) > \mathrm{Im} f^2 > \cdots$ and $\langle e \rangle < \mathrm{Ker}(f) < \mathrm{Ker}(f^2) < \cdots$. Since G satisfies both the ascending chain condition and the descending chain condition then $\mathrm{Im}(f^k) = \mathrm{Im}(f^h)$ and $\mathrm{Ker}(f^j) = \mathrm{Ker}(f^n)$ for all $k \geq n$, for some $n \in \mathbb{N}$.

Suppose $a \in \mathrm{Ker}(f^n) \cap \mathrm{Im}(f^n)$. Then $a = f^n(b)$ for some $b \in G$ (since $a \in \mathrm{Im}(f^n)$) and so $f^{2n}(b) = f^n(f^n(b)) = f^n(a) = e$ (since $a \in \mathrm{Ker}(f^n)$). Consequently, $b \in \mathrm{Ker}(f^{2n}) = \mathrm{Ker}(f^n)$ and so $a = f^n(b) = e$. Therefore $\mathrm{Ker}(f^n) \cap \mathrm{Im}(f^n) = \langle e \rangle$.

Lemma II.3.5 (continued)

Lemma II.3.5. If G is a group that satisfies both the ascending and descending chain conditions on normal subgroups an df is a normal endomorphism of G, then for some $n \geq 1$, we have $G = \text{Ker}(f^n) \times \text{Im}(f^n)$.

Proof (continued). For any $c \in G$ with $f^n(c) \in \text{Im}(f^n) = \text{Im}(f^{2n})$ we have $f^n(c) = f^{2n}(d)$ for some $d \in G$. Thus $f^n(cf^n(d^{-1})) = f^n(c)f^n(f^n(d^{-1})) - f^n(c)f^{2n}(d^{-1}) = f^n(c)(f^{2n}(d))^{-1} = f^n(c)(f^n(c))^{-1} = e$ and hence $c = f^n(d^{-1}) \in \text{Ker}(f^n)$. Since $c = (cf^n(d^{-1}))(f^n(d))$ and c is any element of G, then $G = \text{Ker}(f^n)\text{Im}(f^n)$. By Corollary I.8.7, $G \cong \text{Ker}(f^n) \times \text{Im}(f^n)$. □

Lemma II.3.5 (continued)

Lemma II.3.5. If G is a group that satisfies both the ascending and descending chain conditions on normal subgroups an df is a normal endomorphism of G, then for some $n \geq 1$, we have $G = \text{Ker}(f^n) \times \text{Im}(f^n)$.

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Corollary II.3.6

Corollary II.3.6. If G is an indecomposable group that satisfies both the ascending and descending chain conditions on normal subgroups and f is a normal endomorphism of G, then either f is nilpotent of f is an automorphism.

Proof. By Lemma II.3.5, there is $n \in \mathbb{N}$ such that $G \cong \text{Ker}(f^n) \times \text{Im}(f^n)$. Since G is indecomposable then either $\text{Ker}(f^n) = \langle e \rangle$ or $\text{Im}(f^n) = \langle e \rangle$.

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Proof. By Lemma II.3.5, there is $n \in \mathbb{N}$ such that $G \cong \text{Ker}(f^n) \times \text{Im}(f^n)$. Since G is indecomposable then either $\text{Ker}(f^n) = \langle e \rangle$ or $\text{Im}(f^n) = \langle e \rangle$. If $\text{Im}(f^n) = \langle e \rangle$ then (by definition) f is nilpotent. If $\text{Ker}(f^n) = \langle e \rangle$ then $\text{Ker}(f) = \langle e \rangle$ (since $\langle e \rangle < \text{Ker}(f) < \text{Ker}(f^2) < \cdots$). So by Theorem I.2.3(i), f is a monomorphism (one to one) and by Lemma II.2.4 (the second claim) f is an automorphism. □

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Corollary II.3.7

Corollary II.3.7. Let G (where $G \neq \langle e \rangle$) be an indecomposable group that satisfies both the ascending and descending chain conditions on normal subgroups. If $f_1, f_2, \ldots, f_n$ are normal nilpotent epimorphisms of G such that $f_{i_1} + f_{i_2} + \cdots + f_{i_r}$ (where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$) is an epimorphism, then $f_1 + f_2 + \cdots + f_n$ is nilpotent.

Proof. By Exercise III.3.8(c), if the sum of two normal endomorphisms is itself an endormorphism, then the sum is normal. Induction implies that this holds for any finite sum of normal endormorphisms. Since each $f_{i_1} + f_{i_2} + \cdots + f_{i_r}$ is an endomorphisms by hypothesis, then Exercise II.3.8(c) implies that $f_{i_1} + f_{i_2} + \cdots + f_{i_r}$ is a normal endomorphism. So we prove the corollary for $n = 2$ and then the general result will follow by induction.

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Consider $f_1 + f_2$, a normal endomorphism of G. ASSUME $f_1 + f_2$ is not nilpotent, then by Corollary II.3.6 it is an automorphism of G.

Corollary II.3.7

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Consider $f_1 + f_2$, a normal endomorphism of G. ASSUME $f_1 + f_2$ is not nilpotent, then by Corollary II.3.6 it is an automorphism of G.

Corollary II.3.7 (continued 1)

Proof (continued). So $f_1 + f_2 : G \to G$ has an inverse g which is also an automorphism of G (by Exercise I.2.15(a), the set of automorphisms of G form a group $\mathrm{Aut}(G)$). Then $g^{-1} = f_1 + f_2$ and for all $a, b \in G$ we have

$$\begin{aligned} g(aba^{-1}) &= g(ag^{-1}(b')a^{-1}) \text{ since } b' = g(b) \text{ for some unique } b' \in G \\ &= g(g^{-1}(ab'a^{-1}) \text{ since } g^{-1} = f_1 + f_2 \text{ is normal} \\ &= ab'a^{-1} = ag(b)a^{-1} \end{aligned}$$

and so g is normal. If we define $g_1 = f_1 \circ g = f_1 g$ and $g_2 = f_2 \circ g = f_2 g$ then $g_1 + g_2 = f_1 g + f_2 g = (f_1 + f_2)g = 1_G$ (because for any $a \in G$, let $b = g(a)$ so $a = g^{-1}(b)$, we have

$$\begin{aligned} (g_1 + g_2)(a) &= g_1(a)g_2(a) \text{ by the definition of } g - 1 + g - 2 \\ &= (f_1 \circ g)(a)\,(f_2 \circ g)(a) = f_1(g(a))f_2(g(a)) \\ &= f_1(b)f - 2(b) = (f_1 + f_2)(b) \text{ by the definition of } f_1 + f_2 \\ &= g^{-1}(b)g^{-1}(g(a)) = a.) \end{aligned}$$

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Proof (continued). So $f_1 + f_2 : G \to G$ has an inverse g which is also an automorphism of G (by Exercise I.2.15(a), the set of automorphisms of G form a group $\mathrm{Aut}(G)$). Then $g^{-1} = f_1 + f_2$ and for all $a, b \in G$ we have

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$$\begin{aligned} (g_1 + g_2)(a) &= g_1(a)g_2(a) \text{ by the definition of } g - 1 + g - 2 \\ &= (f_1 \circ g)(a)\,(f_2 \circ g)(a) = f_1(g(a))f_2(g(a)) \\ &= f_1(b)f - 2(b) = (f_1 + f_2)(b) \text{ by the definition of } f_1 + f_2 \\ &= g^{-1}(b)g^{-1}(g(a)) = a.) \end{aligned}$$

Corollary II.3.7 (continued 2)

Proof (continued). So for all $x \in G$,
$x^{-1} = (g_1 + g_2)(x^{-1}) = g_1(x^{-1})g_2(x^{-1})$ (by the definition of $g_1 + g_2$).
Hence

$$\begin{aligned} x &= (g_1(x^{-1}g_2(x^{-1}))^{-1} = (g_2(x^{-1}))^{-1}(g_1(x^{-1}))^{-1} \\ &= g_2(x)g_1(x) \text{ by Exercise I.2.1} \\ &= (g - 2 + g_1)(x) \text{ by the definition of } g_2 + g_1, \end{aligned}$$

so $g_2 + g_1 = 1_G$. Therefore $g_1 + g_2 = g_2 + g_1 = 1_G$ and so $g_1(g_1 + g_2) = g - 11_G = 1_G g_1 = (g_1 + g_2)g_1$ and so $g_1 g_2 = g_2 g_1$ (because for any $a \in G$, since g_1 is a homomorphism, we have $g_1(g_1 + g_2)g_1(a) = g_1(g_1(a)g_2(a)) = g_1(g_1(a))g_a(g_2(a))$ and $(g_1 + g_2)g_1(a) = g_1(g_1(a))g_2(g_1(a))$ by the definition of $g_1 + g_2$, and so $g_1(g_1(a))g_1(g_2(a)) = g_1(g_1(a))g_2(g_1(a))$ and multiplying both sides of this by $g_1(g_1(a))^{-1}$ we have $g_1(g_2(a)) - g_2(g_1(a))$, hence $g_1 g_2 = g_2 g_1$).

Corollary II.3.7 (continued 2)

Proof (continued). So for all $x \in G$,
$x^{-1} = (g_1 + g_2)(x^{-1}) = g_1(x^{-1})g_2(x^{-1})$ (by the definition of $g_1 + g_2$).
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$g_1(g_1 + g_2) = g - 11_G = 1_G g_1 = (g_1 + g_2)g_1$ and so $g_1 g_2 = g_2 g_1$
(because for any $a \in G$, since g_1 is a homomorphism, we have
$g_1(g_1 + g_2)g_1(a) = g_1(g_1(a)g_2(a)) = g_1(g_1(a))g_a(g_2(a))$ and
$(g_1 + g_2)g_1(a) = g_1(g_1(a))g_2(g_1(a))$ by the definition of $g_1 + g_2$, and so
$g_1(g_1(a))g_1(g_2(a)) = g_1(g_1(a))g_2(g_1(a))$ and multiplying both sides of this
by $g_1(g_1(a))^{-1}$ we have $g_1(g_2(a)) - g_2(g_1(a))$, hence $g_1 g_2 = g_2 g_1$).

Corollary II.3.7 (continued 3)

Proof (continued). In Exercise II.3.C it is shown by induction that

$$(g_1 + g_2)^m = \sum_{i=0}^{m} c_i g_1^i g_2^{m-i}$$

where $c_i \in \mathbb{N}$ are the binomial coefficients $c_i = \binom{m}{i} = \frac{m!}{i!(m-i)!}$, to be encountered in Section III.1 in Theorem III.1.6 in the setting of rings. Here, $c_i h$ means $h + h + \cdots + h$ (c_i summands). Since each f_i is nilpotent by hypothesis then $\text{Ker}(f_i) \neq \{e\}$ (or else $\text{Ker}(f_i^n) = \{e\}$ for all $n \in \mathbb{N}$ and f is not nilpotent), so for $g_i = f_i \circ g = f_i g$, where $i \in \{1, 2\}$, we have $\text{Ker}(g_i) = \text{Ker}(f_i g) \neq \{e\}$ and so by Theorem I.2.3(i), g_i is not a monomorphism (not one to one) and hence g_i is not an automorphism as shown above and f_i is a normal endomorphism by hypothesis, then by Exercise II.3.8(a), $g_i = f_i g$ is normal.

Corollary II.3.7 (continued 3)

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Corollary II.3.7 (continued 3)

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Corollary II.3.7 (continued 4)

Proof (continued). Define $n = \max\{n_1, n_2\}$ and choose m large enough that $m/2 \geq n$. Then for $i = 0, 1, \ldots, m$, either i or $m - i$ is greater than or equal to $m/2 \geq n$. For such m we have

$$\begin{aligned}
(g_1 + g_2)^m(a) &= \left(\sum_{i=0}^{m} c_i g_1^i g_2^{m-i}\right)(a) \text{ the sums are in} \\
&\quad \text{the group of functions from } G \text{ to } G \\
&= \prod_{i=0}^{m} (g_1^i(g_2^{m-i}(a)))^{c_i} \text{ by the definition of function sum} \\
&\quad \text{in the group of functions mapping } G \to G \text{ of} \\
&\quad \text{Exercise II.3.B and the notation for } c_i h \text{ and} \\
&\quad \text{the product of functions means composition} \\
&= \prod_{i=0}^{m} e^{c_i} \text{ since either } i \text{ or } m - i \text{ is } \geq n \\
&= e. \qquad (*)
\end{aligned}$$

Corollary II.3.7 (continued 4)

Proof (continued). Define $n = \max\{n_1, n_2\}$ and choose m large enough that $m/2 \geq n$. Then for $i = 0, 1, \ldots, m$, either i or $m - i$ is greater than or equal to $m/2 \geq n$. For such m we have

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\end{aligned}$$

Corollary II.3.7 (continued 5)

Corollary II.3.7. Let G (where $G \neq \langle e \rangle$) be an indecomposable group that satisfies both the ascending and descending chain conditions on normal subgroups. If $f_1, f_2, \ldots, f_n$ are normal nilpotent epimorphisms of G such that $f_{i_1} + f_{i_2} + \cdots + f_{i_r}$ (where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$) is an epimorphism, then $f_1 + f_2 + \cdots + f_n$ is nilpotent.

Proof (continued). But since we showed above that $g_1 + g_2 = 1_G$ then we must have for all $m \in \mathbb{N}$ that

$$(g_1 + g_2)^m = 1_G \qquad (**)$$

(since the exponent means function composition). By hypothesis $G \neq \langle e \rangle$, so there is $a \in G$ with $a \neq e$. We now have $(g_1 + g_2)^m(a) = e$ by $(*)$ and $(g_1 + g_2)^m(a) = a$ by $(**)$, a CONTRADICTION. So the assumption that $f_1 + f_2$ is not nilpotent is false, and hence $f_1 + f_2$ is nilpotent. The general result now holds by induction, as described above. □

Corollary II.3.7 (continued 5)

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Theorem II.3.8. The Krull-Schmidt Theorem

Theorem II.3.8. (The Krull-Schmidt Theorem)
Let G be a group that satisfies both the ascending and descending chain conditions on normal subgroups. If $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s$ and $G = H_1 \times^i H_2 \times^i \cdots \times^i H_t$ with each G_i, H_j indecomposable, then $s = t$ and after reindexing, $G_i \cong H_i$ for every i and for each $r < t$,

$$G = G_1 \times^i G_2 \times^i \cdots \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t.$$

Proof. We start with the hypothesis that $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s$.

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$$G = G_1 \times^i G_2 \times^i \cdots \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t.$$

Proof. We start with the hypothesis that $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s$.

Let $P(0)$ be the statement: "$G = H_1 \times^i H_2 \times^i \cdots \times^i H_t$."

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Theorem II.3.8. (The Krull-Schmidt Theorem)
Let G be a group that satisfies both the ascending and descending chain conditions on normal subgroups. If $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s$ and $G = H_1 \times^i H_2 \times^i \cdots \times^i H_t$ with each G_i, H_j indecomposable, then $s = t$ and after reindexing, $G_i \cong H_i$ for every i and for each $r < t$,

$$G = G_1 \times^i G_2 \times^i \cdots \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t.$$

Proof. We start with the hypothesis that $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s$.

Let $P(0)$ be the statement: "$G = H_1 \times^i H_2 \times^i \cdots \times^i H_t$." For $1 \leq r \leq \min\{s, t\}$, let $P(r)$ be the statement: "There is a reindexing of $H_1, H_2, \ldots, H_t$ such that $G_i \cong H_i$ for $i = 1, 2, \ldots, r$ and $G = G_1 \times^i G_2 \times^i \cdots \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t$ (or $G = G_1 \times^i G_2 \times^i \cdots \times^i G_t$ if $r = t$)." We use induction to prove that $P(r)$ holds for all r such that $0 \leq r \leq \min\{s, t\}$.

Theorem II.3.8. The Krull-Schmidt Theorem

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Let G be a group that satisfies both the ascending and descending chain conditions on normal subgroups. If $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s$ and $G = H_1 \times^i H_2 \times^i \cdots \times^i H_t$ with each G_i, H_j indecomposable, then $s = t$ and after reindexing, $G_i \cong H_i$ for every i and for each $r < t$,

$$G = G_1 \times^i G_2 \times^i \cdots \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t.$$

Proof. We start with the hypothesis that $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s$.

Let $P(0)$ be the statement: "$G = H_1 \times^i H_2 \times^i \cdots \times^i H_t$." For $1 \leq r \leq \min\{s, t\}$, let $P(r)$ be the statement: "There is a reindexing of $H_1, H_2, \ldots, H_t$ such that $G_i \cong H_i$ for $i = 1, 2, \ldots, r$ and $G = G_1 \times^i G_2 \times^i \cdots \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t$ (or $G = G_1 \times^i G_2 \times^i \cdots \times^i G_t$ if $r = t$)." We use induction to prove that $P(r)$ holds for all r such that $0 \leq r \leq \min\{s, t\}$.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 1)

Proof(continued). $P(0)$ is true by hypothesis. Suppose $P(r-1)$ is true; that is, "After some reindexing $G_i \cong H_i$ for $i = 1, 2, \ldots, r-1$ and $G = G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i H_r \times^i H_{r-1} \times^i \cdots \times^i H_t$." Let $\pi_1, \pi_2, \ldots, \pi_s$ be the canonical epimorphism associated with the internal direct product $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s$ (so that $\pi_i : G \to G_i$). Let $\pi_1', \pi_2', \ldots, \pi_i'$ be the canonical epimorphism associated with the internal direct product $G = G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i H_r \times^i \cdots \times^i H_t$ (so that $\pi_i : G \to G_i$ for $1 \le i \le r-1$ and $\pi_i : G \to H_i$ for $r \le i \le t$). Let λ_i be the inclusion map sending G_i into G and let λ_i' be the inclusion map sending the ith factor of $G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i H_r \times^i H_{r+1} \times^i \cdots \times^i H_t$ into G. For each i let $\varphi_i = \lambda_i \pi_i : G \to G$ and let $\psi_i = \lambda_i' \pi_i' : G \to G$ (i.e., φ_i and ψ_i are compositions; notice that the λ_i's and λ_i''s are necessary since π_i maps G to G_i, not G).

Theorem II.3.8. The Krull-Schmidt Theorem (continued 1)

Proof(continued). $P(0)$ is true by hypothesis. Suppose $P(r-1)$ is true; that is, "After some reindexing $G_i \cong H_i$ for $i = 1, 2, \ldots, r-1$ and $G = G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i H_r \times^i H_{r-1} \times^i \cdots \times^i H_t$." Let $\pi_1, \pi_2, \ldots, \pi_s$ be the canonical epimorphism associated with the internal direct product $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s$ (so that $\pi_i : G \to G_i$). Let $\pi_1', \pi_2', \ldots, \pi_i'$ be the canonical epimorphism associated with the internal direct product $G = G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i H_r \times^i \cdots \times^i H_t$ (so that $\pi_i : G \to G_i$ for $1 \le i \le r-1$ and $\pi_i : G \to H_i$ for $r \le i \le t$). Let λ_i be the inclusion map sending G_i into G and let λ_i' be the inclusion map sending the ith factor of $G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i H_r \times^i H_{r+1} \times^i \cdots \times^i H_t$ into G. For each i let $\varphi_i = \lambda_i \pi_i : G \to G$ and let $\psi_i = \lambda_i' \pi_i' : G \to G$ (i.e., φ_i and ψ_i are compositions; notice that the λ_i's and λ_i''s are necessary since π_i maps G to G_i, not G).

Theorem II.3.8. The Krull-Schmidt Theorem (continued 2)

Proof(continued). We claim that we have the following nine identities:

$$\begin{array}{ccc} \varphi_i|_{G_1} = 1_{G_i} & \varphi_i\varphi_i = \varphi_i & \varphi_i\varphi_j = 0_G \text{ for } i \neq j \\ \psi_1 + \psi_2 + \cdots + \psi_t = 1_G & \psi_i\psi_i = \psi_i & \psi_i\psi_j = 0 \text{ for } i \neq j \\ \mathrm{Im}(\varphi_i) = G_i & \mathrm{Im}(\psi_i) = G_i \text{ for } i < r & \mathrm{Im}(\psi_i) = H_i \text{ for } i \geq r. \end{array}$$

We leave the proofs of these claims to Exercise II.3.D. Now for $i < r$ we have for any $x \in G$ that

$$\begin{aligned} \varphi_r\psi_i &= \varphi_r(\psi_i(x)) \\ &= \varphi_r(1_{G_i}(\psi_i(x))) \text{ since } \mathrm{Im}(\psi_i) = F_i \text{ for } i < r \\ &= \varphi_r(\varphi_i(\psi_i(x))) \text{ since } \mathrm{Im}(\psi_i) = G_i \text{ for } i < r \text{ and } \varphi_i|_{G_i} - 1_{G_i} \\ &= (\varphi_r\varphi_i)(\psi_i(x)) \text{ since function composition is associative} \\ &= 0_G(\psi_i(x)) \text{ since } \varphi_i\varphi_j = 0_G \text{ for } i \neq j \\ &= e. \end{aligned}$$

Therefore, $\varphi_y\psi_i = 0_G$ for $i < r$.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 2)

Proof(continued). We claim that we have the following nine identities:

$$\begin{array}{ccc} \varphi_i|_{G_1} = 1_{G_i} & \varphi_i\varphi_i = \varphi_i & \varphi_i\varphi_j = 0_G \text{ for } i \neq j \\ \psi_1 + \psi_2 + \cdots + \psi_t = 1_G & \psi_i\psi_i = \psi_i & \psi_i\psi_j = 0 \text{ for } i \neq j \\ \operatorname{Im}(\varphi_i) = G_i & \operatorname{Im}(\psi_i) = G_i \text{ for } i < r & \operatorname{Im}(\psi_i) = H_i \text{ for } i \geq r. \end{array}$$

We leave the proofs of these claims to Exercise II.3.D. Now for $i < r$ we have for any $x \in G$ that

$$\begin{aligned} \varphi_r\psi_i &= \varphi_r(\psi_i(x)) \\ &= \varphi_r(1_{G_i}(\psi_i(x))) \text{ since } \operatorname{Im}(\psi_i) = F_i \text{ for } i < r \\ &= \varphi_r(\varphi_i(\psi_i(x))) \text{ since } \operatorname{Im}(\psi_i) = G_i \text{ for } i < r \text{ and } \varphi_i|_{G_i} - 1_{G_i} \\ &= (\varphi_r\varphi_i)(\psi_i(x)) \text{ since function composition is associative} \\ &= 0_G(\psi_i(x)) \text{ since } \varphi_i\varphi_j = 0_G \text{ for } i \neq j \\ &= e. \end{aligned}$$

Therefore, $\varphi_y\psi_i = 0_G$ for $i < r$.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 3)

Proof(continued). These identities give

$$\begin{aligned} \varphi_r &= \varphi_r 1_G \\ &= \varphi_r(\psi_1 + \psi_2 + \cdots + \psi_t) \text{ since } \psi_1 + \psi_2 + \cdots + \psi_t = 1_G \\ &= \varphi_r\psi_1 + \varphi_r\psi_2 + \cdots + \varphi_r\psi_t \end{aligned}$$

with the last inequality holding because for $g_1 g_2 \cdots g_t \in G = G_1 \times^i G_2 \times^i \cdots \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t$ we have

$$\begin{aligned} & \varphi_r(\psi_1 + \psi_2 + \cdots + \psi_t)(g_1 g_2 \cdots g_t) \\ = \; & \varphi_r(\psi_1(g_1 g_2 \cdots g_t)\psi_2(g_1 g_2 \cdots g_t) \cdots \psi_t(g_1 g_2 \cdots g_t)) \\ & \text{by the definition of } \psi_1 + \psi_2 + \cdots + \psi_t \\ = \; & \varphi_r(\psi_1(g_1 g - 2 \cdots g_t))\varphi_r(\psi_2(g_1 g_2 \cdots g_t)) \cdots \varphi_r(\psi_t(g_1 g_2 \cdots g_t)) \\ & \text{since } \varphi_r \text{ is a homomorphism} \end{aligned}$$

Theorem II.3.8. The Krull-Schmidt Theorem (continued 3)

Proof(continued). These identities give

$$\begin{aligned} \varphi_r &= \varphi_r 1_G \\ &= \varphi_r(\psi_1 + \psi_2 + \cdots + \psi_t) \text{ since } \psi_1 + \psi_2 + \cdots + \psi_t = 1_G \\ &= \varphi_r\psi_1 + \varphi_r\psi_2 + \cdots + \varphi_r\psi_t \end{aligned}$$

with the last inequality holding because for $g_1g_2\cdots g_t \in G = G_1 \times^i G_2 \times^i \cdots \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t$ we have

$$\begin{aligned} & \varphi_r(\psi_1 + \psi_2 + \cdots + \psi_t)(g_1g_2\cdots g_t) \\ = \; & \varphi_r(\psi_1(g_1g_2\cdots g_t)\psi_2(g_1g_2\cdots g_t)\cdots\psi_t(g_1g_2\cdots g_t)) \\ & \text{by the definition of } \psi_1 + \psi_2 + \cdots + \psi_t \\ = \; & \varphi_r(\psi_1(g_1g - 2\cdots g_t))\varphi_r(\psi_2(g_1g_2\cdots g_t))\cdots\varphi_r(\psi_t(g_1g_2\cdots g_t)) \\ & \text{since } \varphi_r \text{ is a homomorphism} \end{aligned}$$

Theorem II.3.8. The Krull-Schmidt Theorem (continued 4)

Proof(continued).

$$
\begin{aligned}
& \varphi_r(\psi_1 + \psi_2 + \cdots + \psi_t)(g_1 g_2 \cdots g_t) \\
= \ & ee \cdots e \varphi_r(\psi) r(g_1 g_2 \cdots g_t)) \varphi_{r+1}(g_1 g_2 \cdots g_t)) \cdots \varphi_r(\psi_t(g_1 g_2 \cdots g_t)) \\
& \text{since } \varphi_r \psi_i = 0_G \text{ as shown in the previous paragraph} \\
= \ & (\varphi_2 \psi_r + \varphi_r \psi_{r+1} + \cdots + \varphi_r \psi_t)(g_1 g_2 \cdots g_t) \\
& \text{by the definition of } \varphi_2 \psi_r + \varphi_r \psi_{r+1} + \cdots + \varphi_r \psi_t.
\end{aligned}
$$

Since φ_r and ψ_i are normal endomorphisms (since $\text{Im}(\varphi_r) = G_r \triangleleft G$, $\text{Im}(\psi_i) = G_i \triangleleft G$ if $i < r$, and $\text{Im}(\psi_i) = H_i \triangleleft G$ is $i \geq r$) then by Exercise II.3.8(a,) $\varphi_r \psi_i$ is a normal endomorphism. By Exercise II.3.9, every sum of distinct $\varphi_r \psi_i$ is a normal endomorphism.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 4)

Proof(continued).

$$\begin{aligned} & \varphi_r(\psi_1 + \psi_2 + \cdots + \psi_t)(g_1 g_2 \cdots g_t) \\ = \quad & ee \cdots e\varphi_r(\psi)r(g_1 g_2 \cdots g_t))\varphi_{r+1}(g_1 g_2 \cdots g_t)) \cdots \varphi_r(\psi_t(g_1 g_2 \cdots g_t)) \\ & \text{since } \varphi_r \psi_i = 0_G \text{ as shown in the previous paragraph} \\ = \quad & (\varphi_2 \psi_r + \varphi_r \psi_{r+1} + \cdots + \varphi_r \psi_t)(g_1 g_2 \cdots g_t) \\ & \text{by the definition of } \varphi_2 \psi_r + \varphi_r \psi_{r+1} + \cdots + \varphi_r \psi_t. \end{aligned}$$

Since φ_r and ψ_i are normal endomorphisms (since $\mathrm{Im}(\varphi_r) = G_r \lhd G$, $\mathrm{Im}(\psi_i) = G_i \lhd G$ if $i < r$, and $\mathrm{Im}(\psi_i) = H_i \lhd G$ is $i \geq r$) then by Exercise II.3.8(a,) $\varphi_r \psi_i$ is a normal endomorphism. By Exercise II.3.9, every sum of distinct $\varphi_r \psi_i$ is a normal endomorphism. Now $\varphi_r|_{G_r}$ is a (normal) automorphism of G_r and by Exercise II.3.6(b), since G satisfies both the ACC and the DCC then $G_r < G$ also does. ASSUME that normal endomorphism $\varphi_r \psi_j|_{G_r}$ are nilpotent for all j with $r \leq j \leq t$.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 4)

Proof(continued).

$$\begin{aligned} & \varphi_r(\psi_1+\psi_2+\cdots+\psi_t)(g_1g_2\cdots g_t) \\ = \; & ee\cdots e\varphi_r(\psi)r(g_1g_2\cdots g_t))\varphi_{r+1}(g_1g_2\cdots g_t))\cdots\varphi_r(\psi_t(g_1g_2\cdots g_t)) \\ & \text{since } \varphi_r\psi_i = 0_G \text{ as shown in the previous paragraph} \\ = \; & (\varphi_2\psi_r+\varphi_r\psi_{r+1}+\cdots+\varphi_r\psi_t)(g_1g_2\cdots g_t) \\ & \text{by the definition of } \varphi_2\psi_r+\varphi_r\psi_{r+1}+\cdots+\varphi_r\psi_t. \end{aligned}$$

Since φ_r and ψ_i are normal endomorphisms (since $\text{Im}(\varphi_r) = G_r \triangleleft G$, $\text{Im}(\psi_i) = G_i \triangleleft G$ if $i < r$, and $\text{Im}(\psi_i) = H_i \triangleleft G$ is $i \geq r$) then by Exercise II.3.8(a,) $\varphi_r\psi_i$ is a normal endomorphism. By Exercise II.3.9, every sum of distinct $\varphi_r\psi_i$ is a normal endomorphism. Now $\varphi_r|_{G_r}$ is a (normal) automorphism of G_r and by Exercise II.3.6(b), since G satisfies both the ACC and the DCC then $G_r < G$ also does. ASSUME that normal endomorphism $\varphi_r\psi_j|_{G_r}$ are nilpotent for all j with $r \leq j \leq t$.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 5)

Proof(continued). Since every sum of distinct $\varphi_r\psi_j|_{G_r}$ is a normal endomorphism, then by Corollary II.3.7 the sum $(\varphi_r\psi_r + \varphi_r\psi_{r+1} + \cdots + \varphi_r\psi_t)|_{G_r} = \varphi_r|_{G_r}$ is nilpotent, a CONTRADICTION to the fact hat $\varphi_r|_{G_r} = 1_{G_2}$. So the assumption that $\varphi_r\psi_j|_{G_r}$ is nilpotent for all j with $r \leq j \leq t$. So for some j with $r \leq j \leq t$ we have $\varphi_r\psi_j|_{G_r}$ is not nilpotent. By Corollary II.3.6, $\varphi_r\psi_j|_{G_r}$ is therefore an automorphism of G_r. So for every $n \in \mathbb{N}$, $(\varphi_r\psi_j)^{n+1}|_{G_r}$ is also an automorphism of G_r.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 5)

Proof(continued). Since every sum of distinct $\varphi_r\psi_j|_{G_r}$ is a normal endomorphism, then by Corollary II.3.7 the sum $(\varphi_r\psi_r + \varphi_r\psi_{r+1} + \cdots + \varphi_r\psi_t)|_{G_r} = \varphi_r|_{G_r}$ is nilpotent, a CONTRADICTION to the fact hat $\varphi_r|_{G_r} = 1_{G_2}$. So the assumption that $\varphi_r\psi_j|_{G_r}$ is nilpotent for all j with $r \le j \le t$. So for some j with $r \le j \le t$ we have $\varphi_r\psi_j|_{G_r}$ is not nilpotent. By Corollary II.3.6, $\varphi_r\psi_j|_{G_r}$ is therefore an automorphism of G_r. So for every $n \in \mathbb{N}$, $(\varphi_r\psi_j)^{n+1}|_{G_r}$ is also an automorphism of G_r. Now for all $n \in \mathbb{N}$,

$$(\varphi_r\psi_j)^{n+1} = \underbrace{(\varphi_r\psi_j)(\varphi_r\psi_j)\cdots(\varphi_r\psi_j)}_{n+1 \text{ "factors"}}$$

$$= \varphi_r \underbrace{(\varphi_r\psi_j)(\varphi_r\psi_j)\cdots(\varphi_r\psi_j)}_{n \text{ "factors"}} \psi_j = \varphi_r(\varphi_r\psi_j)^n\psi_j.$$

Theorem II.3.8. The Krull-Schmidt Theorem (continued 5)

Proof(continued). Since every sum of distinct $\varphi_r\psi_j|_{G_r}$ is a normal endomorphism, then by Corollary II.3.7 the sum $(\varphi_r\psi_r + \varphi_r\psi_{r+1} + \cdots + \varphi_r\psi_t)|_{G_r} = \varphi_r|_{G_r}$ is nilpotent, a CONTRADICTION to the fact hat $\varphi_r|_{G_r} = 1_{G_2}$. So the assumption that $\varphi_r\psi_j|_{G_r}$ is nilpotent for all j with $r \le j \le t$. So for some j with $r \le j \le t$ we have $\varphi_r\psi_j|_{G_r}$ is not nilpotent. By Corollary II.3.6, $\varphi_r\psi_j|_{G_r}$ is therefore an automorphism of G_r. So for every $n \in \mathbb{N}$, $(\varphi_r\psi_j)^{n+1}|_{G_r}$ is also an automorphism of G_r. Now for all $n \in \mathbb{N}$,

$$(\varphi_r\psi_j)^{n+1} = \underbrace{(\varphi_r\psi_j)(\varphi_r\psi_j)\cdots(\varphi_r\psi_j)}_{n+1 \text{ ``factors''}}$$

$$= \varphi_r \underbrace{(\varphi_r\psi_j)(\varphi_r\psi_j)\cdots(\varphi_r\psi_j)}_{n \text{ ``factors''}} \psi_j = \varphi_r(\varphi_r\psi_j)^n\psi_j.$$

Next, $\psi_j\varphi_r : G \to G$ is a normal endomorphism (by Exercise II.3.8(a)) and $\psi_j\varphi_r|H_j : H_j \to H_j$ (both ψ_j and φ_r are defined on all of G and $\mathrm{Im}(\psi_j) = H_j$ since $j \ge r$).

Theorem II.3.8. The Krull-Schmidt Theorem (continued 5)

Proof(continued). Since every sum of distinct $\varphi_r\psi_j|_{G_r}$ is a normal endomorphism, then by Corollary II.3.7 the sum $(\varphi_r\psi_r + \varphi_r\psi_{r+1} + \cdots + \varphi_r\psi_t)|_{G_r} = \varphi_r|_{G_r}$ is nilpotent, a CONTRADICTION to the fact hat $\varphi_r|_{G_r} = 1_{G_2}$. So the assumption that $\varphi_r\psi_j|_{G_r}$ is nilpotent for all j with $r \le j \le t$. So for some j with $r \le j \le t$ we have $\varphi_r\psi_j|_{G_r}$ is not nilpotent. By Corollary II.3.6, $\varphi_r\psi_j|_{G_r}$ is therefore an automorphism of G_r. So for every $n \in \mathbb{N}$, $(\varphi_r\psi_j)^{n+1}|_{G_r}$ is also an automorphism of G_r. Now for all $n \in \mathbb{N}$,

$$(\varphi_r\psi_j)^{n+1} = \underbrace{(\varphi_r\psi_j)(\varphi_r\psi_j)\cdots(\varphi_r\psi_j)}_{n+1 \text{ ``factors''}}$$

$$= \varphi_r \underbrace{(\varphi_r\psi_j)(\varphi_r\psi_j)\cdots(\varphi_r\psi_j)}_{n \text{ ``factors''}} \psi_j = \varphi_r(\varphi_r\psi_j)^n\psi_j.$$

Next, $\psi_j\varphi_r : G \to G$ is a normal endomorphism (by Exercise II.3.8(a)) and $\psi_j\varphi_r|H_j : H_j \to H_j$ (both ψ_j and φ_r are defined on all of G and $\mathrm{Im}(\psi_j) = H_j$ since $j \ge r$).

Theorem II.3.8. The Krull-Schmidt Theorem (continued 6)

Proof(continued). ASSUME $\psi_j\varphi_r|_{H_j}$ is nilpotent, say $(\psi_j\varphi_r)^n(h) = e$ for all $h \in H_j$. Since $G_r \neq \langle e\rangle$ (because G_r is indecomposable by hypothesis and so $G_r \neq \langle e\rangle$ by the definition of "indecomposable"), then there is some $g \in G_r$ with $g \neq e$. By the induction hypothesis, $G = G_1 \times^i G_2 \times^i \cdots G_{r-1} \times^i H_r \times^i H_{r+1} \times^i \cdots \times^i H_t$, so $g = g_1 g_2 \cdots g_{r-1} h_r h_{r+1} \cdots h_t$ and

$$(\varphi_r\psi_j)^{n+1}(g) = \varphi_r(\psi_j\varphi_r)^n\psi_j(g) = \varphi_r(\psi_j\varphi_r)^n h_i = \varphi_r(e) = e.$$

But then $g \in \mathrm{Ker}((\varphi_r\psi_j)^{n+1})$ and so $\mathrm{Ker}((\varphi_r\psi_j)^{n+1}) \neq \{e\}$ and hence $(\varphi_r\psi_j)^{n+1}|_{G_r}$ is not a monomorphism (one to one) by Theorem I.2.3(i), and so $(\varphi_r\psi_j)^{n+1}|_{G_r}$ is not an automorphism of G_r, a CONTRADICTION. So the assumption that $\psi_j\varphi_r|_{H_j}$ is nilpotent is false.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 6)

Proof(continued). ASSUME $\psi_j\varphi_r|_{H_j}$ is nilpotent, say $(\psi_j\varphi_r)^n(h) = e$ for all $h \in H_j$. Since $G_r \neq \langle e \rangle$ (because G_r is indecomposable by hypothesis and so $G_r \neq \langle e \rangle$ by the definition of "indecomposable"), then there is some $g \in G_r$ with $g \neq e$. By the induction hypothesis, $G = G_1 \times^i G_2 \times^i \cdots G_{r-1} \times^i H_r \times^i H_{r+1} \times^i \cdots \times^i H_t$, so $g = g_1 g_2 \cdots g_{r-1} h_r h_{r+1} \cdots h_t$ and

$$(\varphi_r\psi_j)^{n+1}(g) = \varphi_r(\psi_j\varphi_r)^n\psi_j(g) = \varphi_r(\psi_j\varphi_r)^n h_i = \varphi_r(e) = e.$$

But then $g \in \text{Ker}((\varphi_r\psi_j)^{n+1})$ and so $\text{Ker}((\varphi_r\psi_j)^{n+1}) \neq \{e\}$ and hence $(\varphi_r\psi_j)^{n+1}|_{G_r}$ is not a monomorphism (one to one) by Theorem I.2.3(i), and so $(\varphi_r\psi_j)^{n+1}|_{G_r}$ is not an automorphism of G_r, a CONTRADICTION. So the assumption that $\psi_j\varphi_r|_{H_j}$ is nilpotent is false. Now $H_j < G$ satisfies both the ACC and the DCC (by Exercise II.3.6(b), since G satisfies both) and $\psi_j\varphi_r|_{G_r}$ is a normal endomorphism (because $\psi_j\varphi_r$ is a normal endomorphism on G as shown above), then by Corollary II.3.6, $\psi_j\varphi_r|_{H_j}$ is an automorphism of H_j.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 6)

Proof(continued). ASSUME $\psi_j\varphi_r|_{H_j}$ is nilpotent, say $(\psi_j\varphi_r)^n(h) = e$ for all $h \in H_j$. Since $G_r \neq \langle e \rangle$ (because G_r is indecomposable by hypothesis and so $G_r \neq \langle e \rangle$ by the definition of "indecomposable"), then there is some $g \in G_r$ with $g \neq e$. By the induction hypothesis, $G = G_1 \times^i G_2 \times^i \cdots G_{r-1} \times^i H_r \times^i H_{r+1} \times^i \cdots \times^i H_t$, so $g = g_1 g_2 \cdots g_{r-1} h_r h_{r+1} \cdots h_t$ and

$$(\varphi_r\psi_j)^{n+1}(g) = \varphi_r(\psi_j\varphi_r)^n\psi_j(g) = \varphi_r(\psi_j\varphi_r)^n h_i = \varphi_r(e) = e.$$

But then $g \in \text{Ker}((\varphi_r\psi_j)^{n+1})$ and so $\text{Ker}((\varphi_r\psi_j)^{n+1}) \neq \{e\}$ and hence $(\varphi_r\psi_j)^{n+1}|_{G_r}$ is not a monomorphism (one to one) by Theorem I.2.3(i), and so $(\varphi_r\psi_j)^{n+1}|_{G_r}$ is not an automorphism of G_r, a CONTRADICTION. So the assumption that $\psi_j\varphi_r|_{H_j}$ is nilpotent is false. Now $H_j < G$ satisfies both the ACC and the DCC (by Exercise II.3.6(b), since G satisfies both) and $\psi_j\varphi_r|_{G_r}$ is a normal endomorphism (because $\psi_j\varphi_r$ is a normal endomorphism on G as shown above), then by Corollary II.3.6, $\psi_j\varphi_r|_{H_j}$ is an automorphism of H_j.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 7)

Proof(continued). Now $\varphi_r(H_j) \subset G$ and $\text{Im}(\psi_j \varphi_r|_{H_j}) = H_j$ so that $\psi_j|_{G_r} : G_r \to H_j$ is an isomorphism (and similarly $\varphi_r|_{H_j} : H_j \to G_r$ is an isomorphism). Reindex the H's such that H_j "moves into the rth slot" and becomes H_r so that $G_r \cong H_r$. Then $G_i \cong H_i$ for $i = 1, 2, \ldots, r$ and the first half of claim $P(r)$ holds.

We now need to show that $G = G_1 \times^i G_2 \times^i \cdots \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t$ and $s = t$ By the induction hypothesis $G = G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i H_r \times^i H_{r+1} \times^i \cdots \times^i H_t$. We have the subgroup of G

$$\begin{aligned} & \langle G_1, G_2, \ldots, G_{r-1}, H_{r+1}, H_{r+1}, \ldots, H_t \rangle \\ = \; & G_1 G_2 \cdots G_{r-1} H_{r+1} H_{r+1} \cdots H_t \\ & \text{by "an easily proved generalization of Theorem I.5.3" (see page 61)} \\ = \; & G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i H_{r+1} \times^i H_{r+1} \times^i \cdots \times^i H_t \\ & \text{by the definition of internal direct product (Definition I.8.8).} \end{aligned}$$

Theorem II.3.8. The Krull-Schmidt Theorem (continued 7)

Proof(continued). Now $\varphi_r(H_j) \subset G$ and $\mathrm{Im}(\psi_j \varphi_r|_{H_j}) = H_j$ so that $\psi_j|_{G_r} : G_r \to H_j$ is an isomorphism (and similarly $\varphi_r|_{H_j} : H_j \to G_r$ is an isomorphism). Reindex the H's such that H_j "moves into the rth slot" and becomes H_r so that $G_r \cong H_r$. Then $G_i \cong H_i$ for $i = 1, 2, \ldots, r$ and the first half of claim $P(r)$ holds.

We now need to show that $G = G_1 \times^i G_2 \times^i \cdots \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t$ and $s = t$ By the induction hypothesis $G = G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i H_r \times^i H_{r+1} \times^i \cdots \times^i H_t$. We have the subgroup of G

$$\begin{aligned} & \langle G_1, G_2, \ldots, G_{r-1}, H_{r+1}, H_{r+1}, \ldots, H_t \rangle \\ = \ & G_1 G_2 \cdots G_{r-1} H_{r+1} H_{r+1} \cdots H_t \\ & \text{ by "an easily proved generalization of Theorem I.5.3" (see page 61)} \\ = \ & G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i H_{r+1} \times^i H_{r+1} \times^i \cdots \times^i H_t \\ & \text{by the definition of internal direct product (Definition I.8.8).} \end{aligned}$$

Theorem II.3.8. The Krull-Schmidt Theorem (continued 8)

Proof(continued). Observe that for $j < r$,

$$\begin{aligned} \psi_r(G_j) &= \psi_r\psi_j(G) \text{ since } \mathrm{Im}(\psi_j) = G_j \text{ for } j < r \\ &= \{e\} \text{ since } \psi_r\psi_j = 0_G \text{ for } j \neq r \end{aligned}$$

and for $j > r$,

$$\begin{aligned} \psi_r(H_j) &= \psi_r\psi_j(G) \text{ since } \mathrm{Im}(\psi_j) = H_j \text{ for } j \geq r \\ &= \{e\} \text{ since } \psi_r\psi_j = 0_G \text{ for } j \neq r. \end{aligned}$$

So $\psi_r(G_1G_2\cdots G_{r-1}H_{r+1}H_{r+2}\cdots H_t) = \{e\}$ because each element of the group is mapped to $\underbrace{ee\cdots e}_{t}$ t. Since $\psi_r|_{G_r} = 1_{G_r}$ is an isomorphism it is one to one (injective) on G_r, then applying ψ_r to $G_r \cap (G_1G_2\cdots G_{r-1}H_{r+1}H_{r+2}\cdots H_t)$ yields only $\{e\}$ and so the only element of this intersection must be e (otherwise, $\psi_r|_{G_r}$ would not be injective).

Theorem II.3.8. The Krull-Schmidt Theorem (continued 8)

Proof(continued). Observe that for $j < r$,

$$\begin{aligned}\psi_r(G_j) &= \psi_r\psi_j(G) \text{ since } \mathrm{Im}(\psi_j) = G_j \text{ for } j < r \\ &= \{e\} \text{ since } \psi_r\psi_j = 0_G \text{ for } j \neq r\end{aligned}$$

and for $j > r$,

$$\begin{aligned}\psi_r(H_j) &= \psi_r\psi_j(G) \text{ since } \mathrm{Im}(\psi_j) = H_j \text{ for } j \geq r \\ &= \{e\} \text{ since } \psi_r\psi_j = 0_G \text{ for } j \neq r.\end{aligned}$$

So $\psi_r(G_1G_2\cdots G_{r-1}H_{r+1}H_{r+2}\cdots H_t) = \{e\}$ because each element of the group is mapped to $\underbrace{ee\cdots e}_{t \text{ t}}$. Since $\psi_r|_{G_r} = 1_{G_r}$ is an isomorphism it is one to one (injective) on G_r, then applying ψ_r to $G_r \cap (G_1G_2\cdots G_{r-1}H_{r+1}H_{r+2}\cdots H_t)$ yields only $\{e\}$ and so the only element of this intersection must be e (otherwise, $\psi_r|_{G_r}$ would not be injective).

Theorem II.3.8. The Krull-Schmidt Theorem (continued 9)

Proof(continued). So by the definition of internal direct product (Definition I.8.8) we have

$$\begin{aligned} G^* &= \langle G_1, G_2, \ldots, G_{r-1}, G_r, H_{r+1}, \ldots, H_t \rangle \\ &= G_1 G_2 \cdots G_r H_{r+1} H_{r+2} \cdots H_t \text{ by "an easily proved} \\ &\quad \text{generalization of Theorem I.5.3" (see page 61)} \\ &= G_1 \times^i G_2 \times^i \cdots \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t \end{aligned}$$

(notice that the order does not matter by Theorem I.5.2(iv)).
Every element of G may be written $g = g_1 g_2 \cdots g_{r-1} h_r h_{r+1} \cdots h_t$ where $g_i \in G$ for $1 \leq i \leq r-1$ and $h_j \in H_j$ for $r \leq j \leq t$ (by Theorem I.8.9; by the induction hypothesis $G = G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i H_r \times^i H_{r+1} \times^i \cdots \times^i H_t$). Define $\theta : G \to G$ as $\theta(g) = g_1 g_2 \cdots g_{r-1} \varphi_r(h_r) h_{r+1} h_{r+2} \cdots h_t$. As observes above (in a "similar" statement) $\varphi_r|_{H_j} : H_j \to G_r$ (where $j = r$) is an isomorphism, so $\text{Im}(H_r) = G_r$, and hence $\text{Im}(\theta) = G^*$.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 9)

Proof(continued). So by the definition of internal direct product (Definition I.8.8) we have

$$\begin{aligned} G^* &= \langle G_1, G_2, \ldots, G_{r-1}, G_r, H_{r+1}, \ldots, H_t \rangle \\ &= G_1 G_2 \cdots G_r H_{r+1} H_{r+2} \cdots H_t \text{ by ``an easily proved} \\ &\quad \text{generalization of Theorem I.5.3'' (see page 61)} \\ &= G_1 \times^i G_2 \times^i \cdots \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t \end{aligned}$$

(notice that the order does not matter by Theorem I.5.2(iv)).
Every element of G may be written $g = g_1 g_2 \cdots g_{r-1} h_r h_{r+1} \cdots h_t$ where $g_i \in G$ for $1 \le i \le r-1$ and $h_j \in H_j$ for $r \le j \le t$ (by Theorem I.8.9; by the induction hypothesis
$G = G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i H_r \times^i H_{r+1} \times^i \cdots \times^i H_t$). Define $\theta : G \to G$ as $\theta(g) = g_1 g_2 \cdots g_{r-1} \varphi_r(h_r) h_{r+1} h_{r+2} \cdots h_t$. As observes above (in a "similar" statement) $\varphi_r|_{H_j} : H_j \to G_r$ (where $j = r$) is an isomorphism, so $\mathrm{Im}(H_r) = G_r$, and hence $\mathrm{Im}(\theta) = G^*$.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 10)

Proof(continued). By Theorem I.8.10, since we consider $1_{G_1}, 1_{G_2}, \ldots, 1_{G_{r-1}}, \varphi_r, 1_{H_{r+1}}, 1_{H_{r+2}}, 1_{H_t}$ are each monomorphisms so θ is a monomorphism. By Exercise II.3.E, θ is normal. Therefore by Lemma II.3.4, θ is an automorphism so that $G = \mathrm{Im}(\theta) = G^* = G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t$ and the second part of the inductive claim $P(r)$ holds, completing the inductive argument.

Theorem II.3.8. The Krull-Schmidt Theorem (continued 10)

Proof(continued). By Theorem I.8.10, since we consider $1_{G_1}, 1_{G_2}, \ldots, 1_{G_{r-1}}, \varphi_r, 1_{H_{r+1}}, 1_{H_{r+2}}, 1_{H_t}$ are each monomorphisms so θ is a monomorphism. By Exercise II.3.E, θ is normal. Therefore by Lemma II.3.4, θ is an automorphism so that $G = \mathrm{Im}(\theta) = G^* = G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t$ and the second part of the inductive claim $P(r)$ holds, completing the inductive argument.

We now must just show that $s = t$. After reindexing, $G_i \cong H_i$ for $0 \leq i \leq \min\{s, t\}$. If $s = \min\{s, t\}$ then $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s = G_1 \times^i G_2 \times^i \cdots \times^i G_s \times^i H_{s+1} \times^i \cdots \times^i H_t$. But none of the G_i, H_i are the trivial group $\langle e \rangle$, so $s = t$. If $t = \min\{s, t\}$ then $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s = G_1 \times^i G_2 \times^i \cdots \times^i G_t$ and again $s = t$. □

Theorem II.3.8. The Krull-Schmidt Theorem (continued 10)

Proof(continued). By Theorem I.8.10, since we consider $1_{G_1}, 1_{G_2}, \ldots, 1_{G_{r-1}}, \varphi_r, 1_{H_{r+1}}, 1_{H_{r+2}}, 1_{H_t}$ are each monomorphisms so θ is a monomorphism. By Exercise II.3.E, θ is normal. Therefore by Lemma II.3.4, θ is an automorphism so that $G = \mathrm{Im}(\theta) = G^* = G_1 \times^i G_2 \times^i \cdots \times^i G_{r-1} \times^i G_r \times^i H_{r+1} \times^i H_{r+2} \times^i \cdots \times^i H_t$ and the second part of the inductive claim $P(r)$ holds, completing the inductive argument.

We now must just show that $s = t$. After reindexing, $G_i \cong H_i$ for $0 \leq i \leq \min\{s,t\}$. If $s = \min\{s,t\}$ then $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s = G_1 \times^i G_2 \times^i \cdots \times^i G_s \times^i H_{s+1} \times^i \cdots \times^i H_t$. But none of the G_i, H_i are the trivial group $\langle e \rangle$, so $s = t$. If $t = \min\{s,t\}$ then $G = G_1 \times^i G_2 \times^i \cdots \times^i G_s = G_1 \times^i G_2 \times^i \cdots \times^i G_t$ and again $s = t$. □