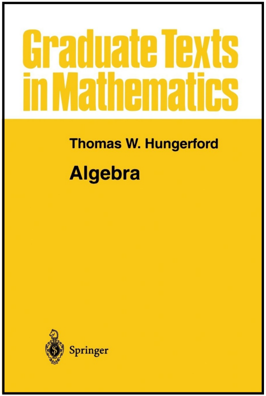
Modern Algebra

Chapter II. The Structure of Groups

II.5. The Sylow Theorems—Proofs of Theorems



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Lemma II.5.1

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Proof. Recall that the orbit of $x \in S$ under action on S is $\overline{x} = \{h \star x \mid h \in H\}$. So an orbit contains exactly one element if and only if $x \in S_0$. Since the orbits represent equivalence classes, then they partition set S so $S = S_0 \cup \overline{x}_1 \cup \overline{x}_2 \cup \cdots \cup \overline{x}_m$ where $|\overline{x}_i| > 1$ for each i. Hence $|S| = |S_0| + |\overline{x}_1| + |\overline{x}_2| + \cdots + |\overline{x}_m|$.

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Theorem II.5.2

Theorem II.5.2. Fraleigh, Theorem 36.3. Cauchy's Theorem.
If G is a finite group whose order is divisible by a prime p, then G contains an element of order p.

Proof. Let S be the set of p-tuples of group elements with product e:

$$S = \{(a_1, a_2, \ldots, a_p) \mid a_i \in G \text{ and } a_1 a_2 \cdots a_p = e\}.$$

Now with $|G| = n$, there are n choices for each of $a_1, a_2, \ldots, a_{p-1}$. But, since the product of the p elements must be e, then $a_p = (a_1 a_2 \cdots a_{p-1})^{-1}$ and so there is only one choice for a_p. So $|S| = n^{p-1}$. Since $p \mid |G|$ (or in the notation, $p \mid n$) then $n \equiv 0 \pmod{p}$ and so $|S| \equiv 0 \pmod{p}$.

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Proof (continued). In a group, if $ab = e$ then $ba = (a^{-1}a)(ba) = a^{-1}(ab)a = a^{-1}ea = e$. In G, since $\underbrace{(a_1 a_2 \cdots a_k)}_{a} \underbrace{(a_{k+1} a_{k+2} \cdots a_p)}_{b} = e$ then $a_{k+1} a_{k+2} \cdots a_p a_1 a_2 \cdots a_k = e$ and so $(a_{k+1}, a_{k+2}, \ldots, a_k) \in S$. Hence this action actually maps $G \times S \to S$ as required by the definition of group action. Next, for $e = 0 \in \mathbb{Z}_p$, we have for $x \in S$ that $0 \star x = x$, satisfying the first condition of group action (Definition II.4.1). Now for $k, k' \in \mathbb{Z}_p$ we have

$$(k + k') \star (a_1, a_2, \ldots, a_p) = (a_{1+k+k'}, a_{2+k+k'}, \ldots, a_p, a_1, \ldots, a_{k+k'})$$

$$= k \star (a_{1+k'}, a_{2+k'}, \ldots, a_p, a_1, \ldots, a_{k'}) = k \star (k' \star (a_1, a_2, \ldots, a_p))$$

(where the indices are reduced as appropriate). So the second condition of the definition of group action is also satisfied. Therefore this is actually an example of group action.

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Theorem II.5.2 (continued 2)

Theorem II.5.2. Fraleigh, Theorem 36.3. Cauchy's Theorem.
If G is a finite group whose order is divisible by a prime p, then G contains an element of order p.

Proof (continued). Now $S_0 = \{x \in S \mid k \star x = x \text{ for all } k \in \mathbb{Z}_p\}$ so $(a_1, a_2, \ldots, a_p) \in S_0$ if and only if $a_1 = a_2 = \cdots = a_p$. Next $(e, e, \ldots, e) \in S_0$ so $|S_0| \neq 0$. By Lemma II.5.1 $|S| \equiv |S_0| \pmod{p}$. By above, $|S| \equiv 0 \pmod{p}$, so $|S_0| \equiv 0 \pmod{p}$. Since $|S_0| \neq 0$ then S_0 must contain at least p elements. That is, there exists $a \neq e$ such that $(a, a, \ldots, a) \in S_0 \subseteq S$. By the definition of S, $aa \cdots a = a^p = e$. Since p is prime, it must be that the order of a is p. □

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A finite group G is a p-group if and only if $|G|$ is a power of p.

Proof. If G is a p-group and q is a prime which divides $|G|$, then G contains an element of order q by Cauchy's Theorem (Theorem II.5.2). Since every element of G has order a power of p (by definition of p-group), then $q = p$. So the only prime divisor of $|G|$ is p and $|G|$ is a power of prime p. Conversely, if $|G|$ is a power of prime p then by Lagrange's Theorem (Corollary I.4.6) every element of G is an order dividing this power of p and so every element is of order a power of prime p. □

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Corollary II.5.4

Corollary II.5.4. The center $C(G)$ of a nontrivial finite p-group G contains more than one element.

Proof. Consider the class equation of G (see Note II.4.A):

$$|G| = |C(G)| + \sum [G : C_G(x_i)]$$

where $C_G(x)$ is the centralizer of x:

$$C_G(x) = \{g \in G \mid gxg^{-1} = x\} = \{g \in G \mid gx = xg\}.$$

Since each $[G : C_G(x_i)] > 1$ (by convention, see Note II.4.A) and $[G : C_G(x_i)]$ divides $|G| = p^n$ ($n \geq 1$; by Corollary II.4.4(i)) then p divides each $[G : C_G(x_i)]$.

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Lemma II.5.5. If H is a p-subgroup of a finite group G, then $[N_G(H) : H] \equiv [G : H] \pmod{p}$.

Proof. Recall that $N_G(H)$ is the normalizer of H:

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\} = \{g \in G \mid gH = Hg\}.$$

Let S be the set of left cosets of H in G and let H act on S by left translation. Then

$$|S| = [G : H]. \qquad (*)$$

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Also by the definition of S_0, ($xH \in S_0$) if and only if ($hxH = xH$ for all $h \in H$) if and only if ($x^{-1}hxH = H$ for all $h \in H$) if and only if ($x^{-1}hx \in H$ for all $h \in H$) if and only if ($x^{-1}Hx = H$) if and only if ($xHx^{-1} = H$) if and only if ($x \in N_G(H)$). Therefore $|S_0|$ is the number of cosets xH with $x \in N_G(H)$.

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Lemma II.5.5 (continued)

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Proof (continued). Now $N_G(H)$ is a group (by Theorem II.4.2, where the group action is conjugation) and H is a subgroup of $N_G(H)$. So $[N_G(H) : H]$ is the number of left cosets of H in $N_G(H)$ and hence

$$|S_0| = [N_G(H) : H]. \qquad (**)$$

By Lemma II.5.1, $|S| \equiv |S_0| \pmod{p}$ and so by $(*)$ and $(**)$, $[N_G(H) : H] \equiv [G : H] \pmod{p}$. □

Corollary II.5.6

Corollary II.5.6. Fraleigh Corollary 36.7
If H is a p-subgroup of a finite group G such that p divides $[G : H]$, then $N_G(H) \neq H$.

Proof. Since p divides $[G : H]$ by hypothesis, then $[G : H] \equiv 0 \pmod{p}$. So from Lemma II.5.5, $[N_G(H) : H] \equiv 0 \pmod{p}$. Since $[N_G(H) : H] \geq 1$ ($eH = H$ is one coset of H) then we must have that $[N_G(H) : H]$ is at least p. So $[N_G(H) : H] > 1$ and $N_G(H) \neq H$ (if $N_G(H) = H$ then there is only one coset of H in $N_G(H)$). □

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Theorem II.5.7

Theorem II.5.7. Fraleigh, Theorem 36.8. First Sylow Theorem.
Let G be a group of order $p^n m$ with $n \geq 1$, p prime, and $(p, m) = 1$. Then G contains a subgroup of order p^i for each $1 \leq i \leq n$ and every subgroup of G of order p^i $(i < n)$ is normal in some subgroup of order p^{i+1}.

Proof. Since $p \mid |G|$, G contains an element a (and therefore a subgroup $\langle a \rangle$) of order p by Cauchy's Theorem (Theorem II.5.2). Now perform induction on i and assume that G has a subgroup H of order p^i where $1 \leq i < n$ (so H is a p-subgroup of G by Corollary II.5.3) we now construct a group H_1 of order p^{i+1} where $H_1 < G$ and $H \triangleleft H_1$). Now $[G : H] = |G|/|H|$ by Lagrange's Theorem (Corollary I.4.6) and since $|H| \leq p^{n-1}$ then $p \mid [G : H]$.

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$$1 < |N_G(H)/H| = [N_G(H) : H] \equiv [G : H] \equiv 0 \pmod{p}.$$

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$$1 < |N_G(H)/H| = [N_G(H) : H] \equiv [G : H] \equiv 0 \pmod{p}.$$

Theorem II.5.7

Proof (continued). Hence $p \mid |N_G(H)/H|$ and $N_G(H)/H$ contains an element bH (and a subgroup $\langle bH \rangle$) of order p by Cauchy's Theorem (Theorem II.5.2). By Corollary I.5.12, this group $\langle bH \rangle$ is of the form H_1/H where $H_1 < N_G(H)$ and $H < H_1$ (in the notation of Corollary I.5.12, $\langle bH \rangle < N_G(H)/H = G/N$ and $K = H_1$; so $K = H_1 < G$, $N = H < H_1 = K$ and $K/N = H_1/H$). Since H is normal in $N_G(H)$ and $H_1 < N_G(H)$ then H is normal in H_1. Finally,

$$|H_1| = |H||H_1/H| \text{ by Lagrange's Theorem}$$
$$= p^i p = p^{i+1}.$$

So $H \triangleleft H_1$ and $|H_1| = p^{i+1}$ and the result follows by induction for all appropriate i. □

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$$\begin{aligned} |H_1| &= |H||H_1/H| \text{ by Lagrange's Theorem} \\ &= p^i p = p^{i+1}. \end{aligned}$$

So $H \triangleleft H_1$ and $|H_1| = p^{i+1}$ and the result follows by induction for all appropriate i. □

Corollary II.5.8

Corollary II.5.8. Let G be a group of order $p^n m$ with p prime, $n \geq 1$, and $(p, m) = 1$. Let H be a p-subgroup of G.

(i) H is a Sylow p-subgroup of G if and only if $|H| = p^n$.

(ii) Every conjugate of a Sylow p-subgroup is a Sylow p-subgroup.

(iii) If there is only one Sylow p-subgroup P, then P is normal in G.

Proof. (i) H is a p-subgroup if and only if $|H|$ is some power of p by Corollary II.5.3. By the First Sylow Theorem (Theorem II.5.7), if $|H| = p^i$ for $0 \leq i < n$ then H is not a Sylow p-subgroup. The only possible Sylow p-subgroups are subgroups of order a power of p by Corollary II.5.3, so (by Lagrange's Theorem) if $|H| = p^n$ then H is a maximal p-subgroup and H is a Sylow p-subgroup; conversely, by the First Sylow Theorem, a Sylow p-subgroup must be of order p^n.

Corollary II.5.8

Corollary II.5.8. Let G be a group of order $p^n m$ with p prime, $n \geq 1$, and $(p, m) = 1$. Let H be a p-subgroup of G.

(i) H is a Sylow p-subgroup of G if and only if $|H| = p^n$.

(ii) Every conjugate of a Sylow p-subgroup is a Sylow p-subgroup.

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Proof. (i) H is a p-subgroup if and only if $|H|$ is some power of p by Corollary II.5.3. By the First Sylow Theorem (Theorem II.5.7), if $|H| = p^i$ for $0 \leq i < n$ then H is not a Sylow p-subgroup. The only possible Sylow p-subgroups are subgroups of order a power of p by Corollary II.5.3, so (by Lagrange's Theorem) if $|H| = p^n$ then H is a maximal p-subgroup and H is a Sylow p-subgroup; conversely, by the First Sylow Theorem, a Sylow p-subgroup must be of order p^n.

Corollary II.5.8 (continued)

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Proof (continued). (ii) This follows from Exercise I.5.6 and part (i).

(iii) If there is only one Sylow p-subgroup P, then by (ii) gPg^{-1} is also a Sylow p-subgroup, so it must be that $gPg^{-1} = P$ for all $g \in G$. That is, by Theorem I.5.1 (and definition), P is normal in G. □

Corollary II.5.8 (continued)

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Theorem II.5.9

Theorem II.5.9. Fraleigh, Theorem 36.10. Second Sylow Theorem. If H is a p-subgroup of a finite group G, and P is any Sylow p-subgroup of G, then there exists $x \in G$ such that $H < xPx^{-1}$. In particular, any two Sylow p-subgroups of G are conjugate.

Proof. Let S be the set of left cosets of P in G and let H act on S by left translation. Now $S_0 = \{xP \in S \mid h(xP) = xP \text{ for all } h \in H\}$ and $[G : P] = |S| \equiv |S_0| \pmod{p}$ by Lemma II.5.1. But $p \nmid [G : P]$ since $[G : P] = |G|/|P| = m$ (where $(m, p) = 1$). So $|S_0| \neq 0$ and there exists $xP \in S_0$. Now $(xP \in S_0)$ if and only if ($hxP = xP$ for all $x \in H$ (by the definition of S_0)) if and only if ($x^{-1}hxP = P$ for all $h \in H$) if and only if $(x^{-1}Hx < P)$ if and only if $(H < xPx^{-1})$, giving the first claim.

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If H is a Sylow p-subgroup, then $|H| = |P|$ by Corollary II.5.8(i). Also, $|P| = |xPx^{-1}|$ by Corollary II.5.8(ii), so $|H| = |xPx^{-1}|$ and it must be that $H = xPx^{-1}$ (since $H < xPx^{-1}$ by above) and so two Sylow p-subgroups P and H must be conjugates. □

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Theorem II.5.10

Theorem II.5.10. Fraleigh, Theorem 36.11. Third Sylow Theorem. If G is a finite group and p a prime, then the number of Sylow p-subgroups of G divides $|G|$ and is of the form $kp + 1$ for some $k \geq 0$.

Proof. By the Second Sylow Theorem (Theorem II.5.9) any two Sylow p-subgroups are conjugate, so if P is a Sylow p-subgroup then the number of conjugates of P is the number of Sylow p-subgroups. But by Corollary II.4.4(iii) the number of conjugates of P in G is $[G : N_G(P)]$ and this is a divisor of $|G|$.

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Let S be the set of all Sylow p-subgroups of G and let P act on S by conjugation. Then $Q \in S_0 = \{Q \in S \mid xQx^{-1} = Q \text{ for all } x \in P\}$ if and only if $P < N_G(Q) = \{x \in G \mid xQx^{-1} = Q\}$. So both P and Q (not necessarily distinct) are Sylow p-subgroups of G and hence of $N_G(Q)$ (since $N_G(Q) < G$) and are therefore conjugate in $N_G(Q)$.

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Let S be the set of all Sylow p-subgroups of G and let P act on S by conjugation. Then $Q \in S_0 = \{Q \in S \mid xQx^{-1} = Q \text{ for all } x \in P\}$ if and only if $P < N_G(Q) = \{x \in G \mid xQx^{-1} = Q\}$. So both P and Q (not necessarily distinct) are Sylow p-subgroups of G and hence of $N_G(Q)$ (since $N_G(Q) < G$) and are therefore conjugate in $N_G(Q)$.

Theorem II.5.10 (continued)

Theorem II.5.10. Fraleigh, Theorem 36.11. Third Sylow Theorem. If G is a finite group and p a prime, then the number of Sylow p-subgroups of G divides $|G|$ and is of the form $kp + 1$ for some $k \geq 0$.

Proof (continued). Since Q is normal in $N_G(Q)$ (by the definition of $N_G(Q)$, the normalizer of Q in G) then every conjugate of Q in $N_G(Q)$ equals Q and so $P = Q$. Therefore $S_0 = \{P\}$. By Lemma II.5.1, $|S| \equiv |S_0| \equiv 1 \pmod{p}$. Hence $|S| = kp + 1$ for some $k \geq 0$. □

Theorem II.5.11

Theorem II.5.11. If P is a Sylow p-subgroup of a finite group G, then $N_G(N_G(P)) = N_G(P)$.

Proof. Every conjugate of P is a Sylow p-subgroup of G by the Second Sylow Theorem (Theorem II.5.9). Every conjugate of P is a Sylow p-subgroup of any subgroup of G that contains it by Corollary II.5.8(ii). Since P is normal in $N = N_G(P) = \{x \in G \mid xPx^{-1} = P\}$, then P is the only Sylow p-subgroup of N by the Second Sylow Theorem (Theorem II.5.9; all Sylow p-subgroups of N must be conjugates, but any conjugate of P in N equals P).

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Proof. Every conjugate of P is a Sylow p-subgroup of G by the Second Sylow Theorem (Theorem II.5.9). Every conjugate of P is a Sylow p-subgroup of any subgroup of G that contains it by Corollary II.5.8(ii). Since P is normal in $N = N_G(P) = \{x \in G \mid xPx^{-1} = P\}$, then P is the only Sylow p-subgroup of N by the Second Sylow Theorem (Theorem II.5.9; all Sylow p-subgroups of N must be conjugates, but any conjugate of P in N equals P). Therefore, $x \in N_G(N_G(P)) = N_G(N)$ if and only if $xNx^{-1} = N$ by the definition of normalizer and this implies that $xPx^{-1} < N$ since $P < N$, and so xPx^{-1} is a Sylow p-subgroup of N by Corollary II.5.8(ii). Since P is the only Sylow p-subgroup of N then $P = xPx^{-1}$ and so $x \in N_G(P) = N$. Therefore $N_G(N_G(P)) \subseteq N_G(P)$.

Theorem II.5.11

Theorem II.5.11. If P is a Sylow p-subgroup of a finite group G, then $N_G(N_G(P)) = N_G(P)$.

Proof. Every conjugate of P is a Sylow p-subgroup of G by the Second Sylow Theorem (Theorem II.5.9). Every conjugate of P is a Sylow p-subgroup of any subgroup of G that contains it by Corollary II.5.8(ii). Since P is normal in $N = N_G(P) = \{x \in G \mid xPx^{-1} = P\}$, then P is the only Sylow p-subgroup of N by the Second Sylow Theorem (Theorem II.5.9; all Sylow p-subgroups of N must be conjugates, but any conjugate of P in N equals P). Therefore, $x \in N_G(N_G(P)) = N_G(N)$ if and only if $xNx^{-1} = N$ by the definition of normalizer and this implies that $xPx^{-1} < N$ since $P < N$, and so xPx^{-1} is a Sylow p-subgroup of N by Corollary II.5.8(ii). Since P is the only Sylow p-subgroup of N then $P = xPx^{-1}$ and so $x \in N_G(P) = N$. Therefore $N_G(N_G(P)) \subseteq N_G(P)$.

Theorem II.5.11 (continued)

Theorem II.5.11. If P is a Sylow p-subgroup of a finite group G, then $N_G(N_G(P)) = N_G(P)$.

Proof (continued). Now "clearly" the normalizers of any subgroup of G contains all the elements of that subgroup and so $x \in N_G(P)$ implies $x \in N_G(N_G(P))$ and $N_G(P) \subseteq N_G(N_G(p))$.

Hence $N_G(N_G(P)) = N_G(P)$. □