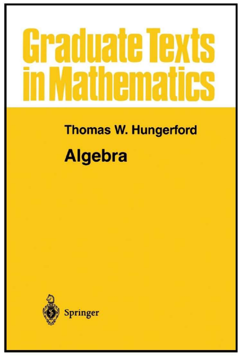
Modern Algebra

Chapter II. The Structure of Groups

II.7. Nilpotent and Solvable Groups—Proofs of Theorems



Proposition II.7.2

Proposition II.7.2. Every finite p-group is nilpotent.

Proof. Let G be the p-group. Then G and all of its nontrivial subgroups are p-groups by the First Sylow Theorem (Theorem II.5.7). Then G and its nontrivial subgroups have nontrivial centers by Corollary II.5.4. Now if $G \neq C_i(G)$ then $G/C_i(G)$ is a nontrivial p-group and $C(G/C_i(G))$ is nontrivial. So $C_i(G)$ is strictly contained in $C_{i+1}(G)$. Since G is finite, $C_n(G)$ must be G for some n and so G is nilpotent. □

Proposition II.7.3

Proposition II.7.3. The direct product of a finite number of nilpotent groups is nilpotent.

Proof. We give the proof for a product of two nilpotent groups and then the general result follows by induction. Let $G = H \times K$. Temporarily assume that $C_i(G) = C_i(H) \times C_i(K)$. Let π_H be the canonical epimorphism mapping $H \to H/C_i(H)$ and let π_K be the canonical epimorphism mapping $K \to K/C_i(K)$. Then $\pi = \pi_H \times \pi_K$ mapping $H \times K \to H/C_i(H) \times K/C_i(K)$ is an epimorphism by Theorem I.8.10. Let ψ be the isomorphism of Corollary I.8.11 which maps $H/C_i(H) \times K/C_i(K) \to (H \times K)/(C_i(H) \times C_i(K))$. Define $\varphi : G \to G/C_i(G)$ as $\varphi = \pi \circ \psi$, so that

$$\begin{aligned} G = H \times K &\overset{\pi}{\to} H/C_i(H) \times K/C_i(K) \\ &\overset{\varphi}{\to} (H \times K)/(C_i(H) \times C_i(K)) \\ &= (H \times K)/C_i(H \times K) \text{ by assumption} \\ &= G/C_i(G). \end{aligned}$$

Proposition II.7.3 (continued 1)

Proposition II.7.3. The direct product of a finite number of nilpotent groups is nilpotent.

Proof (continued). Now if $g \in G$, say $g = (h, k)$, then

$$\begin{aligned} \varphi(g) &= \psi(\pi(g)) = \psi(\pi((h,k)) = \psi(\pi_H(h) \times \pi_K(k)) \\ &= \psi(hC_i(H) \times kC_i(K)) \text{ since } \pi_H \text{ and } \pi_K \text{ are} \\ &\qquad \text{canonical epimorphisms} \\ &= (h,k)C_i(H) \times C_i(K) \text{ by the definition of } \psi \\ &= (h,k)C_i(H \times K) \text{ by assumption} \\ &= gC_i(G). \end{aligned}$$

So $g \in G$ is mapped to coset $gC_i(G) \in G/C_i(G)$ and φ is the canonical epimorphism mapping $G \to G/C_i(G)$.

Proposition II.7.3 (continued 2)

Proposition II.7.3. The direct product of a finite number of nilpotent groups is nilpotent.

Proof (continued). Now

$$\begin{aligned} C(H \times K) &= \{(h,k) \in H \times K \mid (h,k)(x,y) = (x,y)(h,k) \\ &\qquad \text{for all } (x,y) \in H \times K\} \\ &= \{(h,k) \in H \times K \mid (hx,ky) = (xh,yk) \\ &\qquad \text{for all } (x,y) \in H \times K\} \\ &= \{h \in H \mid hx = xh \text{ for all } x \in H\} \\ &\quad \times \{k \in K \mid ky = yk \text{ for all } y \in K\} \\ &= C(H) \times C(K) \qquad (*) \end{aligned}$$

and this holds for *any* H and K.

Proposition II.7.3 (continued 3)

Proposition II.7.3. The direct product of a finite number of nilpotent groups is nilpotent.

Proof (continued). Consequently

$$\begin{aligned} C_{i+1}(G) &= \varphi^{-1}[C(G/C_i(G)] \\ &= \pi^{-1}\varphi^{-1}[C(G/C_i(G))] \\ &= \pi^{-1}\phi^{-1}[C((H \times K)/C_i(H \times H))] \\ &= \pi^{-1}[C(H/C_i(H) \times K/C_i(K))] \text{ by the definition of } \varphi \\ &= \pi^{-1}[C(H/C_i(H)) \times C(K/C_i(K))] \text{ by } (*) \\ &= \pi_H^{-1}[C(H/C_i(H))] \times \pi_K^{-1}[C(K/C_i(K))] \\ &= C_{i+1}(H) \times C_{i+1}(K). \end{aligned}$$

And so by induction on i, $C_i(G) = C_i(H) \times C_i(K)$ for all $i \in \mathbb{N}$.

Proposition II.7.3 (continued 4)

Proposition II.7.3. The direct product of a finite number of nilpotent groups is nilpotent.

Proof (continued). [Notice that we assumed this above simply to have a setting for the mappings π and ψ. The proof by induction on i uses the properties of mappings π and ψ in this setting, but the induction on i is really accomplished by property $(*)$.] Since H and K are nilpotent, then there are $n \in \mathbb{N}$ such that $C_n(H) = H$ and $C_n(K) = K$. So $C_n(G) = C_n)H) \times C_n(K) = H \times K = G$, and G is nilpotent. □

Lemma II.7.4

Lemma II.7.4. If H is a proper subgroup of a nilpotent group G, then H is a proper subgroup of its normalizer $N_G(H)$.

Proof. Recall that $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. Denote $C_0(G) = \{e\}$. Then $C_1(G) = \pi^{-1}(C(G/C_0(G))) = \pi^{-1}(C(G/\{e\})) = C(G)$ as expected. Let n be the largest index such that $C_n(G) < H$ (there is such an n since G is nilpotent and $C_0(G) < H$). Choose $a \in C_{n+1}(G)$ where $a \notin H$. Since $C_{n+1}(G)$ is the inverse image of $C(G/C_n(G))$ under the canonical epimorphism, then $aC_n(G) \in C(G/C_n(G))$ and in $G/C_n(G)$ we have that the cosets satisfy $(aC_n(G))(gC_n(G)) = (gC_n(G))(aC_n(G))$ for all $g \in G$.

Lemma II.7.4 (continued)

Lemma II.7.4. If H is a proper subgroup of a nilpotent group G, then H is a proper subgroup of its normalizer $N_G(H)$.

Proof (continued). Then for every $h \in H$, in $G/C_n(G)$ we have

$$\begin{aligned} C_n(G)ah &= (C_n(G)a)(C_n(G)h) \text{ by the definition of coset multiplication} \\ &= (C_n(G)h)(C_n(G)a) \text{ by the comment above} \\ &\qquad \text{(since } a \in C_{n+1}(G)) \\ &= C_n(G)ha. \end{aligned}$$

Thus $ah = h'ha$ for some $h' \in C_n(G) < H$. Hence $aha^{-1} = h'h \in H$ and so $a \in N_G(H)$. Since $a \notin H$, H is a proper subgroup of $N_G(H)$. □

Proposition II.7.5

Proposition II.7.5. A finite group is nilpotent if and only if it is (isomorphic to) the direct product of its Sylow subgroups.

Proof. If G is the direct product of its Sylow p-subgroups, then by Theorem II.7.2 each of these p-subgroups is nilpotent and by Theorem II.7.3 the product is nilpotent. That is, G is nilpotent.
Now suppose G is nilpotent and let P be a Sylow p-subgroup of G for some prime p. If $P = G$, we are done. So WLOG suppose P is a proper subgroup of G. By Lemma II.7.4, P is a proper subgroup of its normalizer $N_G(P)$. By Theorem II.5.11, $N_G(N_G(P)) = N_G(P)$. With $H = N_G(P)$ in Lemma II.7.4, we see that if $N_G(P)$ is a proper subgroup of G then it would hold that $N_G(P)$ is a proper subgroup of $N_G(N_G(P))$. Since this later case does not hold, it must be that $N_G(P) = G$. Thus P is a normal subgroup of G. By the Second Sylow Theorem (Theorem II.5.9), any other Sylow p-subgroup of G is conjugate with P. But since P is a normal subgroup of G, then it is self conjugate and hence P is the unique Sylow p-subgroup of G.

Proposition II.7.5 (continued)

Proposition II.7.5. A finite group is nilpotent if and only if it is (isomorphic to) the direct product of its Sylow subgroups.

Proof (continued). Let $|G| = p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ where the p_i are distinct primes and each $n_i > 0$. Let $P_1, P_2, \ldots, P_k$ be the corresponding (proper normal) Sylow subgroups of G. Since $|P_i| = p_i^{n_i}$ for each i then each element of P_i is of an order which is a power of p_i by Corollary I.4.6 (Lagrange's Theorem). So $P_i \cap P_j = \{e\}$ for $i \neq j$. By Theorem I.5.3(iv), $xy = yx$ for all $x \in P_i$, $y \in P_j$ and $i \neq j$. This commutivity implies that each element of $P_1P_2\cdots P_{i-1}P_{i+1}\cdots P_k$ has an order dividing $p_1^{n_1}p_2^{n_2}\cdots p_{i-1}^{n_{i-1}}p_{i+1}^{n_{i+1}}\cdots p_k^{n_k}$ (for each i). Consequently $P_i \cap (P_1P_2\cdots P_{i-1}P_{i+1}\cdots P_k) = \{e\}$ (as above) and so by Corollary I.8.7 $P_1P_2\cdots P_k \cong P_1 \times P_2 \times \cdots \times P_k$. Since $|G| = p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k} = |P_1 \times P_2 \times \cdots \times P_k| = |P_1P_2\cdots P_k|$ then we must have $G = P_1P_2\cdots P_k \cong P_1 \times P_2 \times \cdots \times P_k$. □

Theorem II.7.8

Theorem II.7.8. If G is a group, then the commutator subgroup G' is a normal subgroup of G and G/G' is abelian. If N is a normal subgroup of G, then G/N is abelian if and only if N contains G'.

Proof. Let $f : G \to G$ be any automorphism of G. Then by the homomorphism property

$$f(aba^{-1}b^{-1}) = f(a)f(b)f(a^{-1})f(b^{-1}) = f(a)f(b)(f(a))^{-1}(f(b))^{-1} \in G'.$$

By Theorem I.2.8, every element of G' is a finite product of powers of commutators $aba^{-1}b^{-1}$ (where $a, b \in G$) and so $f(G') < G'$. Let f_a be the automorphism of G given by conjugation by a. Then $aG'a^{-1} = f_a(G') < G'$. So every conjugate $aG'a^{-1}$ is a subgroup of G' and by Theorem I.5.1(iv), G' is a normal subgroup of G. Since all $a, b \in G$, we have $a^{-1}, b^{-1} \in G$ and so $a^{-1}b^{-1}(a^{-1})^{-1}(b^{-1})^{-1} = a^{-1}b^{-1}ab \in G'$ and so $a^{-1}b^{-1}abG' = G'$ or $abG' = baG'$. But then by the definition of coset multiplication, $(aG')(bG') = abG' = baG' = (bG')(aG')$ and so coset multiplication is commutative and G/G' is abelian.

Theorem II.7.8 (continued)

Theorem II.7.8. If G is a group, then the commutator subgroup G' is a normal subgroup of G and G/G' is abelian. If N is a normal subgroup of G, then G/N is abelian if and only if N contains G'.

Proof (continued). Let N be a normal subgroup of G. Suppose G/N is abelian. Then $abN = baN$ for all $a, b \in G$. In particular, $a^{-1}b^{-1}N = b^{-1}a^{-1}N$ or $aba^{-1}b^{-1}N = N$ and $aba^{-1}b^{-1} \in N$. Therefore N contains all commutators $aba^{-1}b^{-1}$ $(a, b \in G)$ and $G' < N$. Conversely, suppose $G' < N$. Then for all $a, b \in G$ we have (as above) $a^{-1}, b^{-1} \in G$ and so $a^{-1}b^{-1}ab \in G'$, so $a^{-1}b^{-1}ab \in N$ and $a^{-1}b^{-1}abN = N$ or $abN = baN$ and then G/N is abelian. □

Proposition II.7.10

Proposition II.7.10. Every nilpotent group is solvable.

Proof. By the definition of $C_i(G)$ as the inverse image of $C(G/C_{i-1}(G))$ under the canonical homomorphism π mapping $G \to G/C_{i-1}(G)$, we have $C_i(G)/C_{i-1}(G) = C(G/C_{i-1}(G))$. [Hmmm...]
Now the center of a group is the set of elements of the group which commute with all the elements of G, and so a center is an abelian group. So $C(G/C_{i-1}(G)) = C_i(G)/C_{i-1}(G)$ is abelian. By Theorem II.7.8, $C_{i-1}(G)$ contains $C_i(G)$ for all $i > 1$. Since $C(G)$ is abelian then the commutator subgroup $C_1(G)' = C(G) = \{e\}$ by the comment made after the definition of commutator. Since G is nilpotent then, by definition, for some $n \in \mathbb{N}$ we have $C_n(G) = G$. Therefore (with $i = n$) $C(G/C_{n-1}(G)) = C_n(G)/C_{n-1}(G) = G/C_{n-1}(G)$ is abelian and hence, by Theorem II.7.8, $C_{n-1}(G)$ contains G', or $G^{(1)} = G' < C_{n-1}(G)$.

Proposition II.7.10 (continued)

Proposition II.7.10. Every nilpotent group is solvable.

Proof (continued). Now if A is a subgroup of B then the commutator group A' is a subgroup of the commutator group B'. So

$$G^{(2)} = G^{(1)\prime} < C_{n-1}(G)' < C_{n-2}(G) \text{ (by the above with } i = n-1).$$

Similarly $G^{(3)} < C_{n-2}(G)' < C_{n-3}(G), \ldots, G^{(n-1)} < C_2(G)' < C_1(G)$, and $G^{(n)} < C_1(G)' = C(G) = \{e\}$ since $C(G)$ is abelian. So $G^{(n)} = \{e\}$ and G is solvable. □

Theorem II.7.11

Theorem II.7.11.

(i) Every subgroup and every homomorphic image of a solvable group is solvable.

(ii) If N is a normal subgroup of a group G such that N and G/N are solvable, then G is solvable.

Proof. (i) Let $f : G \to H$ be a homomorphism. Let G' and H' be the commutator subgroups of G and H. By Theorem I.2.8, the elements of G' are finite products of powers of $(aba^{-1}b^{-1})$ where $a, b \in G$. Applying f to such an element produces a finite product of powers of $f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1}$. So $f(G') < H'$. If f is onto (that is, an epimorphism) then $f(G') = H'$. Since $G^{(i)} = (G^{(i-1)})'$, it follows by induction that $f(G^{(i)}) < H^{(i)}$ (with equality if f is an epimorphism) for all $i \in \mathbb{N}$. Suppose f is onto (an epimorphism) and so H is a homomorphic image of G. Also, suppose G is solvable. Then for some n, $\{e\} = f(e) = f(G^{(n)}) = H^{(n)}$ and so H is solvable.

Theorem II.7.11

Theorem II.7.11.

(i) Every subgroup and every homomorphic image of a solvable group is solvable.

(ii) If N is a normal subgroup of a group G such that N and G/N are solvable, then G is solvable.

Proof (i) (continued). Now if H is a subgroup of G, then $H' < G'$ and $H^{(i)} < G^{(i)}$ for all $i \in \mathbb{N}$. If G is solvable then for some n, $\{e\} = G^{(n)}$ and so $H^{(n)} = \{e\}$ and H is solvable.
(ii) Let $f : G \to G/N$ be the canonical homomorphism (or "epimorphism"). Since G/N is solvable by hypothesis, then for some $n \in \mathbb{N}$ we have $f(G^{(n)}) = (G/N)^{(n)}$ as in the proof of (i) and $G^{(n)} = \{e\}$ so $f(G^{(n)}) = (G/N)^{(n)} = \{e\}$. So $G^{(n)} < \mathrm{Ker}(f) = N$. Since $G^{(n)}$ is a subgroup of N, then by part (i) $G^{(n)}$ is solvable. So for some $k \in \mathbb{N}$ we have $(G^{(n)})^{(k)} = \{e\}$ and so $G^{(n+k)} = (G^{(n)})^{(k)} = \{e\}$ and G is solvable. □

Lemma II.7.13

Lemma II.7.13. let N be a normal subgroup of a finite group G and H any subgroup of G.

(i) If H is a characteristic subgroup of N, then H is normal in G.

(ii) Every normal Sylow p-subgroup of G is fully invariant.

(iii) If G is solvable and N is a minimal normal subgroup, then N is an abelian p-group for some prime p.

Proof. (i) Since $aNa^{-1} = N$ for all $a \in G$ because N is hypothesized as normal. So conjugation by a is an automorphism of N. Since H is characteristic in N by hypothesis, then $aHa^{-1} < H$ for all $a \in G$. Hence H is normal in G by Theorem I.5.1(iv).
(ii) This is left as an exercise.
(iii) Consider the commutator subgroup of N, N', which is generated by the set $\{aba^{-1}b^{-1} \mid a, b \in N\}$. By Theorem I.2.8, the elements of N' are finite products of powers of $(aba^{-1}b^{-1})$ where $a, b \in N$.

Lemma II.7.13 (continued)

Lemma II.7.13. let N be a normal subgroup of a finite group G and H any subgroup of G.

(iii) If G is solvable and N is a minimal normal subgroup, then N is an abelian p-group for some prime p.

Proof (iii) (continued). So if $f : G \to N$ is a homomorphism (that is, f is an endomorphism of N) then the image of an element in N' is again a finite product of powers of commutators of elements of N (since $f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1}$). So $f(N') < N'$ and N' is a fully invariant subgroup of N. Since every fully invariant subgroup is characteristic, then N' is a characteristic subgroup of N. By part (i), N' is a normal subgroup of G. Since N is hypothesized to be a minimal normal subgroup of G, then either $N' = \{e\}$ or $N' = N$. Since N is a subgroup of solvable group G then by Theorem II.7.11(i), N is solvable. So $N' \neq N$ (otherwise the chain of derived subgroups of N would be $N > N > N > \cdots$ and N would not be solvable; that is, we would have $N^{(i)} = N$ for all $i \in \mathbb{N}$ and not have $N^{(n)} = \{e\}$ for some $n \in \mathbb{N}$).

Lemma II.7.13 (continued again)

Lemma II.7.13. let N be a normal subgroup of a finite group G and H any subgroup of G.

(iii) If G is solvable and N is a minimal normal subgroup, then N is an abelian p-group for some prime p.

Proof (iii) (continued). Hence $N' = \{e\} \neq N$ and N is a nontrivial abelian group (since any group G is abelian if and only if $G' = \{e\}$—see the note after Definition II.7.9). Let P be a nontrivial Sylow p-subgroup of N for some prime p (which exists by the First Sylow Theorem [Theorem II.5.7]). Since N is abelian then P is normal in N. By part (ii), P is a fully invariant subgroup of N and, since every fully invariant subgroup is characteristic (see the note following the definition of "characteristic subgroup") then P is characteristic in N and so by part (i), P is normal in G. Since N is a minimal normal subgroup of G by hypothesis and P is a nontrivial subgroup of N then $P = N$ and so $|N| = P$ for some prime P. □