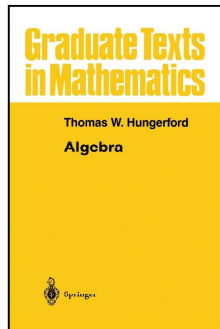


Modern Algebra

Chapter II. The Structure of Groups

II.8. Normal and Subnormal Series—Proofs of Theorems



Proposition II.8.4

Theorem II.8.4.

- (i) Every finite group G has a composition series.
- (ii) Every refinement of a solvable series is a solvable series.
- (iii) A subnormal series is a composition series if and only if it has no proper refinements.

Proof. (i) Let G_1 be a maximal normal subgroup of G ($\{e\}$ is a normal subgroup, so the finite collection of normal subgroups is nonempty and a maximal normal subgroup of a finite group must exist). Then G/G_1 is simple by Corollary I.5.12 (Corollary I.5.12 implies that if $N \triangleleft G$ then every normal subgroup of G/N is of the form H/N where H is a normal subgroup of G which contains N . Therefore, when $G \neq N$, G/N is simple and has no normal subgroups itself if and only if N is a maximal normal subgroup of G .) Let G_2 be a maximal normal subgroup of G_1 , and so on. Since G is finite, this process must end with $G_n = \{e\}$. The series we have created, $G > G_1 > G_2 > \cdots > G_n = \{e\}$ is a composition series since each G_i/G_{i+1} is simple.

Proposition II.8.4(ii)

Theorem II.8.4.

- (i) Every finite group G has a composition series.
- (ii) Every refinement of a solvable series is a solvable series.
- (iii) A subnormal series is a composition series if and only if it has no proper refinements.

Proof. (ii) In a solvable series $G = G_1 > G_2 > \cdots > G_n$, G_i/G_{i+1} is abelian (by definition). If $G_{i+1} \triangleleft H \triangleleft G_i$ is a part of a refinement of a solvable series, then H/G_{i+1} is abelian since H/G_{i+1} is a subgroup of G_i/G_{i+1} . Also, G_i/H is abelian since, by the Third Isomorphism Theorem (Theorem I.5.10), $G_i/H \cong (G_i/G_{i+1})/(H/G_{i+1})$ (a quotient of abelian groups). So the one-step refinement is a solvable series and by induction, any refinement is solvable.

Proposition II.8.4(ii)

Theorem II.8.4.

- (i) Every finite group G has a composition series.
- (ii) Every refinement of a solvable series is a solvable series.
- (iii) A subnormal series is a composition series if and only if it has no proper refinements.

Proof. (iii) Let $G = G_0 > G_1 > \cdots > G_n$ be a subnormal series. If $G_{i+1} \triangleleft H \triangleleft G_i$ where the normal subgroups inclusions are proper inclusions, then H/G_{i+1} is a proper normal subgroup of G_i/G_{i+1} and every proper normal subgroup of G_i/G_{i+1} is of this form by Corollary I.5.12. So the subnormal series has a refinement if and only if such a subgroup H exists. If the subnormal series is a composition series then each G_i/G_{i+1} is simple (by definition) and no such H exists, then G_i/G_{i+1} is simple (by Corollary I.5.12, G_i/G_{i+1} has no normal subgroup without subgroup H) and the subnormal series is a composition series. \square

Proposition II.8.5

Theorem II.8.5. A group G is solvable if and only if it has a solvable series.

Proof. Suppose G is solvable. Then by the definition of “solvable,” in the derived series of commutator subgroups we have $G^{(n)} = \{e\}$ for some $n \in \mathbb{N}$. By Theorem II.7.8, in the series $G > G^{(1)} > G^{(2)} > \dots > G^{(n)} = \{e\}$ we have that $G^{(i+1)}$ is normal in $G^{(i)}$ and $G^{(i)}/G^{(i+1)}$ is abelian. So the series is subnormal (because each subgroup is normal in each previous subgroup) and is also solvable (since the quotient groups are abelian).

Now suppose $G = G_0 > G_1 > \dots > G_n = \{e\}$ is a solvable series. Then G_i/G_{i+1} is abelian (by definition of solvable series) for $0 \leq i \leq n-1$. By Theorem II.7.8, $G_{i+1} > (G_i)'$ for $0 \leq i \leq n-1$; G_i/G_{i+1} abelian implies $(G_i)' > G^{(i+1)}$.

Proposition II.8.5 (continued)

Theorem II.8.5. A group G is solvable if and only if it has a solvable series.

Proof (continued). Since in the derived series of commutator subgroups we have $G > G^{(1)} > G^{(2)} > \dots > G^{(n)}$, then

$$\begin{aligned} G_1 &> G'_0 > G' = G^{(1)} \quad (i = 0) \\ G_2 &> G'_1 > (G^{(1)})' = G^{(2)} \quad (i = 1, i = 0) \\ G_3 &> G'_2 > (G^{(2)})' = G^{(3)} \quad (i = 2, i = 1) \\ &\vdots \\ G_{i+1} &> G'_i > (G^{(i)})' = G^{(i+1)} \\ &\vdots \\ G_n &> G'_{n-1} > (G^{(n-1)})' = G^{(n)}. \end{aligned}$$

But $G_n = \{e\}$ so it must be that $G^{(n)} = \{e\}$ and G is solvable. \square

Proposition II.8.6

Proposition II.8.6. A finite group G is solvable if and only if G has a composition series whose factors are cyclic and of prime order.

Proof. Suppose G has a composition series whose factors are cyclic and each prime order. Then, since cyclic groups are abelian, then the factor groups are abelian and G is solvable. Conversely, assume $G = G_0 > G_1 > \dots > G_n = \{e\}$ is a solvable series for G (so this is a subnormal series of G). If $G_0 \neq G_1$, let H_1 be a maximal normal subgroup of $G = G_0$ which contains G_1 . If $H_1 \neq G_1$, let H_2 be a maximal normal subgroup of H_1 which contains G_1 , and so on. Since G is finite, this gives a series $G = G_0 > H_1 > H_2 > \dots > H_k > G_1$ with each subgroup a maximal normal subgroup of the preceding. Whence each factor is simple by Note A above. Repeat this process for each pair (G_i, G_{i+1}) to get the series refinement $G = N_0 > N_1 > \dots > H_r = \{e\}$ of the original solvable series. By Theorem II.8.4(ii), this refinement series is solvable. So, by definition of solvable series, each factor is abelian.

Proposition II.8.6 (continued)

Proposition II.8.6. A finite group G is solvable if and only if G has a composition series whose factors are cyclic and of prime order.

Proof (continued). So the factors are abelian and simple; but, since every subgroup of an abelian group is normal, then the only simple abelian groups (by Corollary II.2.4) are those of prime order. So each factor group is isomorphic to \mathbb{Z}_p for some p (by Exercise I.4.3). Since the factors are simple, the series is a composition series (by the definition of composition series). So group G has a composition series whose factors are cyclic of prime order. \square

Lemma II.8.8

Lemma II.8.8. If S is a composition series of a group G , then any refinement of S is equivalent to S .

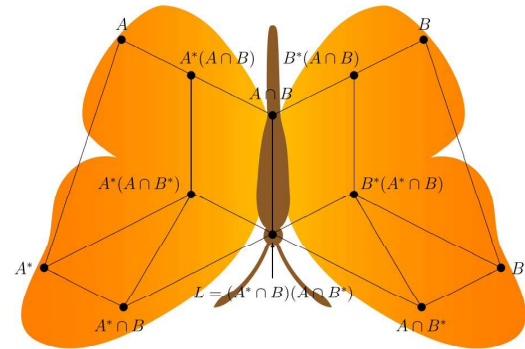
Proof. Let the composition series S be denoted $G = G_0 > G_1 > \dots > G_n = \{e\}$. By Theorem II.8.4(iii) S has no proper refinements. So the only refinements of S are obtained by inserting additional copies of each G_i . Consequently, any refinement of S has exactly the same nontrivial factors as S . So any refinement of S is equivalent to S . □

Lemma II.8.9. Zassenhaus' Lemma/The Butterfly Lemma

Lemma II.8.9. Zassenhaus' Lemma/The Butterfly Lemma.

Let A^* , A , B^* , B be subgroups of a group G such that A^* is normal in A and B^* is normal in B .

- (i) $A^*(A \cap B^*)$ is a normal subgroup of $A^*(A \cap B)$.
- (ii) $B^*(A^* \cap B)$ is a normal subgroup of $B^*(A \cap B)$.
- (iii) $A^*(A \cap B)/A^*(A \cap B^*) \cong B^*(A \cap B)/B^*(A^* \cap B)$.



Lemma II.8.9. Zassenhaus' Lemma (continued 1)

Lemma II.8.9. Zassenhaus' Lemma/The Butterfly Lemma.

Let A^* , A , B^* , B be subgroups of a group G such that A^* is normal in A and B^* is normal in B .

- (i) $A^*(A \cap B^*)$ is a normal subgroup of $A^*(A \cap B)$.
- (ii) $B^*(A^* \cap B)$ is a normal subgroup of $B^*(A \cap B)$.
- (iii) $A^*(A \cap B)/A^*(A \cap B^*) \cong B^*(A \cap B)/B^*(A^* \cap B)$.

Proof. In Theorem I.5.3(i) with $G = B$, $N = B^*$ normal in $G = B$, and $K = A \cap B$ a subgroup of $G = B$, we have that $N \cap K = B^* \cap (A \cap B)$ is a normal subgroup of $K = A \cap B$. Since $M^* \subset B$, we have $A \cap B^* = (A \cap B) \cap B^*$ and so $A \cap B^*$ is normal in $A \cap B$. Similarly (interchanging A and B), $A^* \cap B$ is normal in $A \cap B$. By Exercise I.5.13, $(A^* \cap B) \vee (A \cap B^*)$ (where \vee denotes the join) is a normal subgroup of $A \cap B$. By Theorem I.5.3(iii), $(A^* \cap B) \vee (A \cap B^*) = (A^* \cap B)(A \cap B^*)$, and so $D = (A^* \cap B)(A \cap B^*)$ is a normal subgroup of $A \cap B$. Since $A \cap B < A$ and $A^* \triangleleft G$, then by Theorem I.5.3(iii) (again), $A^* \vee (A \cap B) = A^*(A \cap B)$ is a subgroup of A ; similarly $B^*(A \cap B)$ is a subgroup of B .

Lemma II.8.9. Zassenhaus' Lemma (continued 2)

Proof (continued). In the next paragraph, we will define an epimorphism (onto homomorphism) $f : A^*(A \cap B) \rightarrow (A \cap B)/D$ with kernel $A^*(A \cap B^*)$. Since the kernel of a homomorphism is a normal subgroup (Theorem I.5.5), then this will imply that $A^*(A \cap B^*)$ is normal in $A^*(A \cap B)$. By the First Isomorphism Theorem (Corollary I.5.7) we have that $A^*(A \cap B)/A^*(A \cap B^*) \cong (A \cap B)/D$.

Define $f : A^*(A \cap B) \rightarrow (A \cap B)/D$ as follows. If $a \in A^*$ and $c \in (A \cap B)$ then let $f(ac) = Dc$. Notice that f is well defined since $ac = a_1c_1$ (where $a, a_1 \in A^*$ and $c, c_1 \in A \cap B$) implies

$$\begin{aligned} c_1c^{-1} &= a_1^{-1}a \in (A \cap B) \cap A^* \text{ since } c_1c^{-1} \in A \cap B \text{ and } a_1^{-1}a \in A^* \\ &= A^* \cap B \text{ since } A < A^* \\ &< D. \end{aligned}$$

Whence $Dc_1 = Dc$ since $c_1c^{-1} \in D$. Since $f(ac) = Dc$ and c ranges over all of $A \cap B$ then f is onto $(A \cap B)/D$.

Lemma II.8.9. Zassenhaus' Lemma (continued 3)

Proof (continued). As above, by Theorem I.5.3(iii), $A^*(A \cap B) = A^* \vee (A \cap B) = (A \cap B)A^*$ so for any $c_1 a_2 \in (A \cap B)A^*$ we have that $c_1 a_2 \in A^*(A \cap B)$ and $c_1 a_2 = a_3 c_1$ for some $a_3 \in A^*$. So f is a homomorphism since

$$\begin{aligned} f((a_1 c_1)(a_2 c_2)) &= f(a_1(c_1 a_2)c_2) \\ &= f(a_1 a_3 c_1 c_2) \text{ by above} \\ &= D(c_1 c_2) \\ &= (Dc_1)(Dc_2) \text{ by the definition of coset multiplication} \\ &= f((a_1 c_1))f((a_2 c_2)). \end{aligned}$$

Finally $ac \in \text{Ker}(f)$ if and only if $c \in D$ (that is, if and only if $c = a_1 c_1$ with $a_1 \in A^* \cap B$ and $c_1 \in A \cap B^*$, since $D = (A^* \cap B)(A \cap B^*)$). Hence $ac \in \text{Ker}(f)$ if and only if $ac = a(a_1 c_1) = (aa_1)c_1 \in A^*(A \cap B^*)$. Therefore, $\text{Ker}(f) = A^*(A \cap B^*)$. As commented in the previous paragraph, this implies that $A^*(A \cap B^*)$ is normal in $A^*(A \cap B)$, and (i) follows.

Lemma II.8.9. Zassenhaus' Lemma (continued 4)

Lemma II.8.9. Zassenhaus' Lemma/The Butterfly Lemma.

Let A^* , A , B^* , B be subgroups of a group G such that A^* is normal in A and B^* is normal in B .

- (i) $A^*(A \cap B^*)$ is a normal subgroup of $A^*(A \cap B)$.
- (ii) $B^*(A^* \cap B)$ is a normal subgroup of $B^*(A \cap B)$.
- (iii) $A^*(A \cap B)/A^*(A \cap B^*) \cong B^*(A \cap B)/B^*(A^* \cap B)$.

Proof (continued). A symmetric argument shows that $B^*(A^* \cap B)$ is normal in $B^*(A \cap B)$ and (ii) follows.

In both arguments, as described above, we have

$A^*(A \cap B)/A^*(A \cap B^*) \cong (A \cup B)/D$ and $B^*(A \cap B)/B^*(A^* \cap B) \cong (A \cup B)/D$ by Corollary I.5.7. Therefore, $A^*(A \cap B)/A^*(A \cap B^*) \cong B^*(A \cap B)/B^*(A^* \cap B)$, and (iii) follows. □

Theorem II.8.10. Schreier's Theorem

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Any two subnormal series of a group G have subnormal refinements that are equivalent. Any two normal series of a group G have normal refinements that are equivalent.

Proof. Let $G = G_0 > G_1 > \dots > G_n$ and $G = H_0 > H_1 > \dots > H_m$ be subnormal [normal] series, respectively. Let $G_{n+1} = H_{m+1} = \{e\}$. For $0 \leq i \leq n$ consider the groups

$$\begin{aligned} G_i &= G_{i+1}(G_i \cap H_0) > G_{i+1}(G_i \cap H_1) > \dots > G_{i+1}(G_i \cap H_j) \\ &> G_{i+1}(G_i \cap H_{j+1}) > \dots > G_{i+1}(G_i \cap H_m) \supset G_{i+1}(G_i \cap H_{m+1}) = G_{i+1} \end{aligned}$$

(the subgroup inclusion follows since each $H_j > H_{j+1}$). Since the two series are subnormal [normal] then $G_{i+1} \triangleleft G_i$ and $H_{j+1} \triangleleft H_j$. Applying the Zassenhaus Lemma (Lemma II.8.9) with $A = G_i$, $A^* = G_{i+1}$, $B = H_j$ and $B^* = H_{j+1}$ we have that $A^*(A \cap B^*) = G_{i+1}(G_i \cap H_{j+1})$ is normal in $A^*(A \cap B) = G_{i+1}(G_i \cap H_j)$ for all $0 \leq j \leq m$.

Theorem II.8.10. Schreier's Theorem (continued 1)

Theorem II.8.10. Schreier's Theorem.

Any two subnormal series of a group G have subnormal refinements that are equivalent. Any two normal series of a group G have normal refinements that are equivalent.

Proof (continued). [If the two original series were both normal, then $G_i \cap H_j$ is a normal subgroup of G by Exercise I.5.2, $G_{i+1} \vee (G_i \cap H_j)$ is normal by Exercise I.5.13, and $G_{i+1} \vee (G_i \cap H_j) = G_{i+1}(G_i \cap H_j)$ by Theorem I.5.3(iii). So $G_{i+1}(G_i \cap H_j)$ is a normal subgroup of G and the refinement series we are about to create will be a normal series.] Inserting these groups between G_i and G_{i+1} and denoting $G_{i+1}(G_i \cap H_j)$ by $G(i, j)$ thus gives a subnormal [normal] refinement of the series $G_0 > G_1 > \dots > G_n$:

$$\begin{aligned} G &= G(0, 0) > G(0, 1) \dots > G(0, m) > G(1, 0) = G_1 \\ &> G(1, 1) > G(1, 2) \dots > G(1, m) > G(2, 0) = G_2 \\ &> \dots > G(n-1, 0m) > G(n, 0) > \dots > G(n, m) = G_n \end{aligned}$$

where $G(i, 0) = G_i$.

Theorem II.8.10. Schreier's Theorem (continued 2)

Theorem II.8.10. Schreier's Theorem.

Any two subnormal series of a group G have subnormal refinements that are equivalent. Any two normal series of a group G have normal refinements that are equivalent.

Proof (continued). Note that this refinement has $(n+1)(m+1)$ (not necessarily distinct) terms. A "symmetric argument" (with the G_i 's replaced with the H_i 's) shows that there is a refinement of $G = H_0 > H_1 > \cdots > H_m$ (where $H(i, j) = H_{j+1}(G_1 \cap H_j)$ and $H(0, j) = H_j$):

$$\begin{aligned} G &= H(0, 0) > H(1, 0) < \cdots < H(n, 0) > H(0, 1) = H_1 \\ &> H(1, 1) > H(2, 1) > \cdots > H(n, 1) > H(0, 2) = H_2 \\ &> \cdots > H(n, m-1) > H(0, m) > \cdots > H(n, m) = H_m. \end{aligned}$$

This refinement also has $(n+1)(m+1)$ (not necessarily distinct) terms.

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Theorem II.8.10. Schreier's Theorem (continued 3)

Theorem II.8.10. Schreier's Theorem.

Any two subnormal series of a group G have subnormal refinements that are equivalent. Any two normal series of a group G have normal refinements that are equivalent.

Proof (continued). For each pair (i, j) (with $0 \leq i \leq n$ and $0 \leq j \leq m$) there is, by the Zassenhaus Lemma part(iii) (Lemma I.8.9(iii)) with $A = G_i$, $A^* = G_{i+1}$, $B = H_j$, and $B^* = H_{j+1}$ we have

$$\begin{aligned} \frac{A^*(A \cap B)}{A^*(A \cap B^*)} &= \frac{G_{i+1}(G_i \cap H_j)}{G_{i+1}(G_i \cap H_{j+1})} = \frac{G(i, j)}{G(i, j+1)} \\ &\cong \frac{B^*(A \cap B)}{B^*(A^* \cap B)} = \frac{H_{j+1}(G_i \cap H_j)}{H_{j+1}(G_{i+1} \cap H_j)} = \frac{H(i, j)}{H(i+1, j)}. \end{aligned}$$

So the factors for the two refinements are in a one to one correspondence of isomorphic pairs. That is, the two refinements are equivalent. \square

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Theorem II.8.11. Jordan-Hölder Theorem

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Any two composition series of a group G are equivalent. Therefore every group having a composition series determines a unique list of simple groups.

Proof. By definition, every composition series is a subnormal series. By Schreier's Theorem (Theorem II.8.10) any two composition series have equivalent refinements. But every refinement of a composition series S is equivalent to S by Lemma II.8.8. So if we start with two composition series of G , say S^1 and S^2 , then there are equivalent composition series S_R^1 and S_R^2 where $S_R^1 = S$ and $S_R^2 = S$ (here "=" represents equivalence) and so $S_R^1 = S_R^2$. \square

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