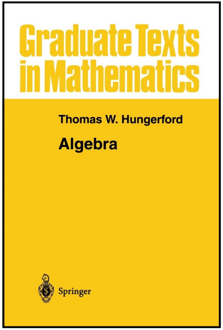
Modern Algebra

Chapter II. The Structure of Groups

II.8. Normal and Subnormal Series—Proofs of Theorems



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(ii) Every refinement of a solvable series is a solvable series.

(iii) A subnormal series is a composition series if and only if it has no proper refinements.

Proof. **(i)** Let G_1 be a maximal normal subgroup of G ($\{e\}$ is a normal subgroup, so the finite collection of normal subgroups is nonempty and a maximal normal subgroup of a finite group must exist).

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Proof. (ii) In a solvable series $G = G_1 > G_2 > \cdots > G_n$, G_i/G_{i+1} is abelian (by definition). If $G_{i+1} \triangleleft H \triangleleft G_i$ is a part of a refinement of a solvable series, then H/G_{i+1} is abelian since H/G_{i+1} is a subgroup of G_i/G_{i+1}.

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$$\begin{aligned}
G_1 &> G_0' > G' = G^{(1)} \ (i = 0)\\
G_2 &> G_1' > (G^{(1)})' = G^{(2)} \ (i = 1, i = 0)\\
G_3 &> G_2' > (G^{(2)})' = G^{(3)} \ (i = 2, i = 1)\\
&\vdots\\
G_{i+1} &> G_i' > (G^{(i)})' = G^{(i+1)}\\
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Proposition II.8.6. A finite group G is solvable if and only if G has a composition series whose factors are cyclic and of prime order.

Proof. Suppose G has a composition series whose factors are cyclic and each prime order. Then, since cyclic groups are abelian, then the factor groups are abelian and G is solvable.

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Conversely, assume $G = G_0 > G_1 > \cdots > G_n = \{e\}$ is a solvable series for G (so this is a subnormal series of G). If $G_0 \neq G_1$, let H_1 be a maximal normal subgroup of $G = G_0$ which contains G_1. If $H_1 \neq G_1$, let H_2 be a maximal normal subgroup of H_1 which contains G_1, and so on.

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Proof (continued). So the factors are abelian and simple; but, since every subgroup of an abelian group is normal, then the only simple abelian groups (by Corollary II.2.4) are those of prime order. So each factor group is isomorphic to $\mathbb{Z}_p$ for some p (by Exercise I.4.3). Since the factors are simple, the series is a composition series (by the definition of composition series). So group G has a composition series whose factors are cyclic of prime order. □

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Lemma II.8.8

Lemma II.8.8. If S is a composition series of a group G, then any refinement of S is equivalent to S.

Proof. Let the composition series S be denoted $G = G_0 > G_1 > \cdots > G_n = \{e\}$. By Theorem II.8.4(iii) S has no proper refinements.

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Lemma II.8.9. Zassenhaus' Lemma/The Butterfly Lemma

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Let A^*, A, B^*, B be subgroups of a group G such that A^* is normal in A and B^* is normal in B.

(i) $A^*(A \cap B^*)$ is a normal subgroup of $A^*(A \cap B)$.

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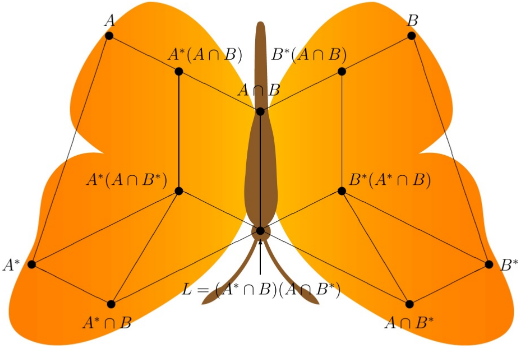
(iii) $A^*(A \cap B)/A^*(A \cap B^*) \cong B^*(A \cap B)/B^*(A^* \cap B)$.



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Proof. In Theorem I.5.3(i) with $G = B$, $N = B^*$ normal in $G = B$, and $K = A \cap B$ a subgroup of $G = B$, we have that $N \cap K = B^* \cap (A \cap B)$ is a normal subgroup of $K = A \cap B$. Since $M^* \subset B$, we have $A \cap B^* = (A \cap B) \cap B^*$ and so $A \cap B^*$ is normal in $A \cap B$. Similarly (interchanging A and B), $A^* \cap B$ is normal in $A \cap B$.

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Lemma II.8.9. Zassenhaus' Lemma (continued 2)

Proof (continued). In the next paragraph, we will define an epimorphism (onto homomorphism) $f : A^*(A \cap B) \to (A \cap B)/D$ with kernel $A^*(A \cap B^*)$. Since the kernel of a homomorphism is a normal subgroup (Theorem I.5.5), then this will imply that $A^*(A \cap B^*)$ is normal in $A^*(A \cap B)$. By the First Isomorphism Theorem (Corollary I.5.7) we have that $A^*(A \cap B)/A^*(A \cap B^*) \cong (A \cap B)/D$.

Lemma II.8.9. Zassenhaus' Lemma (continued 2)

Proof (continued). In the next paragraph, we will define an epimorphism (onto homomorphism) $f : A^*(A \cap B) \to (A \cap B)/D$ with kernel $A^*(A \cap B^*)$. Since the kernel of a homomorphism is a normal subgroup (Theorem I.5.5), then this will imply that $A^*(A \cap B^*)$ is normal in $A^*(A \cap B)$. By the First Isomorphism Theorem (Corollary I.5.7) we have that $A^*(A \cap B)/A^*(A \cap B^*) \cong (A \cap B)/D$.

Define $f : A^*(A \cap B) \to A \cap B)/D$ as follows. If $a \in A^*$ and $c \in (A \cap B)$ then let $f(ac) = Dc$.

Lemma II.8.9. Zassenhaus' Lemma (continued 2)

Proof (continued). In the next paragraph, we will define an epimorphism (onto homomorphism) $f : A^*(A \cap B) \to (A \cap B)/D$ with kernel $A^*(A \cap B^*)$. Since the kernel of a homomorphism is a normal subgroup (Theorem I.5.5), then this will imply that $A^*(A \cap B^*)$ is normal in $A^*(A \cap B)$. By the First Isomorphism Theorem (Corollary I.5.7) we have that $A^*(A \cap B)/A^*(A \cap B^*) \cong (A \cap B)/D$.

Define $f : A^*(A \cap B) \to A \cap B)/D$ as follows. If $a \in A^*$ and $c \in (A \cap B)$ then let $f(ac) = Dc$. Notice that f is well defined since $ac = a_1 c_1$ (where $a, a_1 \in A^*$ and $c, c_2 \in A \cap B$) implies

$$\begin{aligned} c_1 c^{-1} &= a_1^{-1} a \in (A \cap B) \cap A^* \text{ since } c_1 c^{-1} \in A \cap B \text{ and } a_1^{-1} a \in A^* \\ &= A^* \cap B \text{ since } A < A^* \\ &< D. \end{aligned}$$

Whence $Dc_1 = Dc$ since $c_1 c^{-1} \in D$. Since $f(ac) = Dc$ and c ranges over all of $A \cap B$ then f is onto $(A \cap B)/D$.

Lemma II.8.9. Zassenhaus' Lemma (continued 2)

Proof (continued). In the next paragraph, we will define an epimorphism (onto homomorphism) $f : A^*(A \cap B) \to (A \cap B)/D$ with kernel $A^*(A \cap B^*)$. Since the kernel of a homomorphism is a normal subgroup (Theorem I.5.5), then this will imply that $A^*(A \cap B^*)$ is normal in $A^*(A \cap B)$. By the First Isomorphism Theorem (Corollary I.5.7) we have that $A^*(A \cap B)/A^*(A \cap B^*) \cong (A \cap B)/D$.
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Whence $Dc_1 = Dc$ since $c_1c^{-1} \in D$. Since $f(ac) = Dc$ and c ranges over all of $A \cap B$ then f is onto $(A \cap B)/D$.

Lemma II.8.9. Zassenhaus' Lemma (continued 3)

Proof (continued). As above, by Theorem I.5.3(iii), $A^*(A \cap B) = A^* \vee (A \cap B) = (A \cap B)A^*$ so for any $c_1 a_2 \in (A \cap B)A^*$ we have that $c_1 a_2 \in A^*(A \cap B)$ and $c_1 a_2 = a_3 c_1$ for some $a_3 \in A^*$. So f is a homomorphism since

$$\begin{aligned} f((a_1c_1)(a_2c_2)) &= f(a_1(c_1a_2)c_2) \\ &= f(a_1a_3c_1c_2) \text{ by above} \\ &= D(c_1c_2) \\ &= (Dc_1)(Dc_2) \text{ by the definition of coset multiplication} \\ &= f((a_1c_1))f((a_2c_2)). \end{aligned}$$

Lemma II.8.9. Zassenhaus' Lemma (continued 3)

Proof (continued). As above, by Theorem I.5.3(iii), $A^*(A \cap B) = A^* \vee (A \cap B) = (A \cap B)A^*$ so for any $c_1 a_2 \in (A \cap B)A^*$ we have that $c_1 a_2 \in A^*(A \cap B)$ and $c_1 a_2 = a_3 c_1$ for some $a_3 \in A^*$. So f is a homomorphism since

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Finally $ac \in \text{Ker}(f)$ if and only if $c \in D$ (that is, if and only if $c = a_1 c_1$ with $a_1 \in A^* \cap B$ and $c_1 \in A \cap B^*$, since $D = (A^* \cap B)(A \cap B^*)$. Hence $ac \in \text{Ker}(f)$ if and only if $ac = a(a_1 c_1) = (aa_1)c_1 \in A^*(A \cap B^*)$.

Lemma II.8.9. Zassenhaus' Lemma (continued 3)

Proof (continued). As above, by Theorem I.5.3(iii), $A^*(A \cap B) = A^* \vee (A \cap B) = (A \cap B)A^*$ so for any $c_1 a_2 \in (A \cap B)A^*$ we have that $c_1 a_2 \in A^*(A \cap B)$ and $c_1 a_2 = a_3 c_1$ for some $a_3 \in A^*$. So f is a homomorphism since

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Lemma II.8.9. Zassenhaus' Lemma (continued 3)

Proof (continued). As above, by Theorem I.5.3(iii), $A^*(A \cap B) = A^* \vee (A \cap B) = (A \cap B)A^*$ so for any $c_1 a_2 \in (A \cap B)A^*$ we have that $c_1 a_2 \in A^*(A \cap B)$ and $c_1 a_2 = a_3 c_1$ for some $a_3 \in A^*$. So f is a homomorphism since

$$\begin{aligned} f((a_1 c_1)(a_2 c_2)) &= f(a_1(c_1 a_2)c_2) \\ &= f(a_1 a_3 c_1 c_2) \text{ by above} \\ &= D(c_1 c_2) \\ &= (Dc_1)(Dc_2) \text{ by the definition of coset multiplication} \\ &= f((a_1 c_1))f((a_2 c_2)). \end{aligned}$$

Finally $ac \in \text{Ker}(f)$ if and only if $c \in D$ (that is, if and only if $c = a_1 c_1$ with $a_1 \in A^* \cap B$ and $c_1 \in A \cap B^*$, since $D = (A^* \cap B)(A \cap B^*)$. Hence $ac \in \text{Ker}(f)$ if and only if $ac = a(a_1 c_1) = (aa_1)c_1 \in A^*(A \cap B^*)$. Therefore, $\text{Ker}(f) = A^*(A \cap B^*)$. As commented in the previous paragraph, this implies that $A^*(A \cap B^*)$ is normal in $A^*(A \cap B)$, and (i) follows.

Lemma II.8.9. Zassenhaus' Lemma (continued 4)

Lemma II.8.9. Zassenhaus' Lemma/The Butterfly Lemma.
Let A^*, A, B^*, B be subgroups of a group G such that A^* is normal in A and B^* is normal in B.

(i) $A^*(A \cap B^*)$ is a normal subgroup of $A^*(A \cap B)$.

(ii) $B^*(A^* \cap B)$ is a normal subgroup of $B^*(A \cap B)$.

(iii) $A^*(A \cap B)/A^*(A \cap B^*) \cong B^*(A \cap B)/B^*(A^* \cap B)$.

Proof (continued). A symmetric argument shows that $B^*(A^* \cap B)$ is normal in $B^*(A \cap B)$ and (ii) follows.
In both arguments, as described above, we have $A^*(A \cap B)/A^*(A \cap B^*) \cong (A \cup B)/D$ and $B^*(A \cap B)/B^*(A^* \cap B) \cong (A \cup B)/D$ by Corollary I.5.7. Therefore, $A^*(A \cap B)/A^*(A \cap B^*) \cong B^*(A \cap B)/B^*(A^* \cap B)$, and (iii) follows. □

Lemma II.8.9. Zassenhaus' Lemma (continued 4)

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Let A^*, A, B^*, B be subgroups of a group G such that A^* is normal in A and B^* is normal in B.

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(iii) $A^*(A \cap B)/A^*(A \cap B^*) \cong B^*(A \cap B)/B^*(A^* \cap B)$.

Proof (continued). A symmetric argument shows that $B^*(A^* \cap B)$ is normal in $B^*(A \cap B)$ and (ii) follows.
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$A^*(A \cap B)/A^*(A \cap B^*) \cong (A \cup B)/D$ and
$B^*(A \cap B)/B^*(A^* \cap B) \cong (A \cup B)/D$ by Corollary I.5.7. Therefore,
$A^*(A \cap B)/A^*(A \cap B^*) \cong B^*(A \cap B)/B^*(A^* \cap B)$, and (iii) follows. □

Theorem II.8.10. Schreier's Theorem

Theorem II.8.10. Schreier's Theorem.
Any two subnormal series of a group G have subnormal refinements that are equivalent. Any two normal series of a group G have normal refinements that are equivalent.

Proof. Let $G = G_0 > G_1 > \cdots > G_n$ and $G = H_0 > H_1 > \cdots > H_n$ be subnormal [normal] series, respectively. Let $G_{n+1} = H_{m+1} = \{e\}$.

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$$G_i = G_{i+1}(G_i \cap H_0) > G_{i+1}(G_i \cap H_1) > \cdots > G_{i+1}(G_i \cap H_j)$$

$$> G_{i+1}(G_i \cap H_{j+1}) > \cdots G_{i+1}(G_i \cap H_m) \supset G_{i+1}(G_i \cap H_{m+1}) = G_{i+1}$$

(the subgroup inclusion follows since each $H_j > H_{j+1}$). Since the two series are subnormal [normal] then $G_{i+1} \triangleleft G_i$ and $H_{j+1} \triangleleft H_j$.

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(the subgroup inclusion follows since each $H_j > H_{j+1}$). Since the two series are subnormal [normal] then $G_{i+1} \triangleleft G_i$ and $H_{j+1} \triangleleft H_j$. Applying the Zassenhaus Lemma (Lemma II.8.9) with $A = G_i$, $A^* = G_{i+1}$, $B = H_j$ and $B^* = H_{j+1}$ we have that $A^*(A \cap B^*) = G_{i+1}(G_i \cap H_{j+1})$ is normal in $A^*(A \cap B) = G_{i+1}(G_i \cap H_j)$ for all $0 \le j \le m$.

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(the subgroup inclusion follows since each $H_j > H_{j+1}$). Since the two series are subnormal [normal] then $G_{i+1} \triangleleft G_i$ and $H_{j+1} \triangleleft H_j$. Applying the Zassenhaus Lemma (Lemma II.8.9) with $A = G_i$, $A^* = G_{i+1}$, $B = H_j$ and $B^* = H_{j+1}$ we have that $A^*(A \cap B^*) = G_{i+1}(G_i \cap H_{j+1})$ is normal in $A^*(A \cap B) = G_{i+1}(G_i \cap H_j)$ for all $0 \leq j \leq m$.

Theorem II.8.10. Schreier's Theorem (continued 1)

Theorem II.8.10. Schreier's Theorem.
Any two subnormal series of a group G have subnormal refinements that are equivalent. Any two normal series of a group G have normal refinements that are equivalent.

Proof (continued). [If the two original series were both normal, then $G_i \cap H_j$ is a normal subgroup of G by Exercise I.5.2, $G_{i+1} \vee (G_i \cap H_j)$ is normal by Exercise I.5.13, and $G_{i+1} \vee (G_i \cap H_j) = G_{i+1}(G_i \cap H_j)$ by Theorem I.5.3(iii). So $G_{i+1}(G_i \cap H_j)$ is a normal subgroup of G and the refinement series we are about to create will be a normal series.] Inserting these groups between G_i and G_{i+1} and denoting $G_{i+1}(G_i \cap H_j)$ by $G(i,j)$ thus gives a subnormal [normal] refinement of the series $G_0 > G_1 > \cdots > G_n$:

$$G = G(0,0) > G(0,1) \cdots > G(0,m) > G(1,0) = G_1$$
$$> G(1,1) > G(1,2) \cdots > G(1,m) > G(2,0) = G_2$$
$$> \cdots > G(n-1,0m) > G(n,0) > \cdots > G(n,m) = G_n$$

where $G(i,0) = G_i$.

Theorem II.8.10. Schreier's Theorem (continued 1)

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where $G(i,0) = G_i$.

Theorem II.8.10. Schreier's Theorem (continued 2)

Theorem II.8.10. Schreier's Theorem.
Any two subnormal series of a group G have subnormal refinements that are equivalent. Any two normal series of a group G have normal refinements that are equivalent.

Proof (continued). Note that this refinement has $(n+1)(m+1)$ (not necessarily distinct) terms. A "symmetric argument" (with the G_i's replaced with the H_i's) shows that there is a refinement of $G = H_0 > H_1 > \cdots > H_m$ (where $H(i,j) = H_{j+1}(G_1 \cap H_j)$ and $H(0,j) = H_j$):

$$G = H(0,0) > H(1,0) < \cdots > H(n,0) > H(0,1) = H_1$$

$$> H(1,1) > H(2,1) > \cdots > H(n,1) > H(0,2) = H_2$$

$$> \cdots > H(n,m-1) > H(0,m) > \cdots > H(n,m) = H_m.$$

This refinement also has $(n+1)(m+1)$ (not necessarily distinct) terms.

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This refinement also has $(n+1)(m+1)$ (not necessarily distinct) terms.

Theorem II.8.10. Schreier's Theorem (continued 3)

Theorem II.8.10. Schreier's Theorem.
Any two subnormal series of a group G have subnormal refinements that are equivalent. Any two normal series of a group G have normal refinements that are equivalent.

Proof (continued). For each pair (i,j) (with $0 \le i \le n$ and $0 \le j \le m$) there is, by the Zassenhaus Lemma part(iii) (Lemma I.8.9(iii)) with $A = G_i$, $A^* = G_{i+1}$, $B = H_j$, and $B^* = H_{j+1}$ we have

$$\frac{A^*(A \cap B)}{A^*(A \cap B^*)} = \frac{G_{i+1}(G_i \cap H_j)}{G_{i+1}(G_i \cap H_{j+1}} = \frac{G(i,j)}{G(i,j+1)}$$

$$\cong \frac{B^*(A \cap B)}{B^*(A^* \cap B)} = \frac{H_{j+1}(G_i \cap H_j)}{H_{j+1}(G_{i+1} \cap H_j)} = \frac{H(i,j)}{H(i+1,j)}.$$

So the factors for the two refinements are in a one to one correspondence of isomorphic pairs. That is, the two refinements are equivalent. □

Theorem II.8.10. Schreier's Theorem (continued 3)

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Any two subnormal series of a group G have subnormal refinements that are equivalent. Any two normal series of a group G have normal refinements that are equivalent.

Proof (continued). For each pair (i, j) (with $0 \leq i \leq n$ and $0 \leq j \leq m$) there is, by the Zassenhaus Lemma part(iii) (Lemma I.8.9(iii)) with $A = G_i$, $A^* = G_{i+1}$, $B = H_j$, and $B^* = H_{j+1}$ we have

$$\frac{A^*(A \cap B)}{A^*(A \cap B^*)} = \frac{G_{i+1}(G_i \cap H_j)}{G_{i+1}(G_i \cap H_{j+1}} = \frac{G(i,j)}{G(i,j+1)}$$

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So the factors for the two refinements are in a one to one correspondence of isomorphic pairs. That is, the two refinements are equivalent. □

Theorem II.8.11. Jordan-Hölder Theorem

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Any two composition series of a group G are equivalent. Therefore every group having a composition series determines a unique list of simple groups.

Proof. By definition, every composition series is a subnormal series. By Schreier's Theorem (Theorem II.8.10) any two composition series have equivalent refinements.

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Proof. By definition, every composition series is a subnormal series. By Schreier's Theorem (Theorem II.8.10) any two composition series have equivalent refinements. But every refinement of a composition series S is equivalent to S by Lemma II.8.8. So if we start with two composition series of G, say S^1 and S^2, then there are equivalent composition series S_R^1 and S_R^2 where $S_R^1 = S$ and $S_R^2 = S$ (here "$=$" represents equivalence) and so $S_R^1 = S_R^2$. □

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