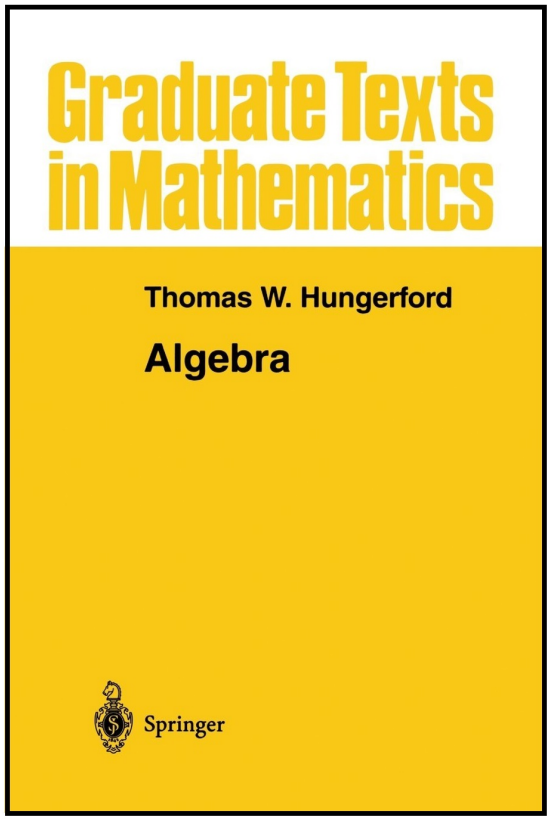
# Modern Algebra

## Chapter III. Rings

III.3. Factorization in Commutative Rings—Proofs of Theorems



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# Theorem III.3.4

**Theorem III.3.4.** Let $p$ and $c$ be nonzero elements in an integral domain $R$.

(i) $p$ is prime if and only if $(p)$ is a nonzero prime ideal.

(ii) $c$ is irreducible if and only if $(c)$ is maximal in the set $S$ of all proper principal ideals of $R$.

(iii) Every prime element of $R$ is irreducible.

(iv) If $R$ is a principal ideal domain, then $p$ is prime if and only if $p$ is irreducible.

(v) Every associate of an irreducible (respectively, prime) element of $R$ is irreducible (respectively, prime).

(vi) The only divisors of an irreducible element of $R$ are its associates and the units of $R$.

# Theorem III.3.4 (continued 1)

**Theorem III.3.4.** Let $p$ and $c$ be nonzero elements in an integral domain $R$.

(i) $p$ is prime if and only if $(p)$ is a nonzero prime ideal.

**Proof. (i)** Let $p$ be prime. Since $R$ is an integral domain then $R$ is commutative with an identity, so $p$ is in the center of $R$ and by Theorem III.2.5(v), $(p) = \{rp \mid r \in R\}$. Let $ab \in (p)$. Then $ab = rp$ for some $r \in R$ and $p \mid ab$. Since $p$ is prime then (by Definition III.3.3), either $p \mid a$ or $p \mid b$. But then either $a = r_1 p$ or $b = r_2 p$ for some $r_1, r_2 \in R$, which implies that either $a \in (p)$ or $b \in (p)$. By Theorem III.2.15, we have that principal ideal $(p)$ is prime.

# Theorem III.3.4 (continued 1)

**Theorem III.3.4.** Let $p$ and $c$ be nonzero elements in an integral domain $R$.

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Let $(p)$ be a nonzero prime ideal. Let $p \mid ab$. Then $ab = rp$ for some $r \in R$ and $ab \in (p)$. Since $R$ is commutative, Theorem III.2.15 implies that either $a \in (p)$ or $b \in (p)$. So, by Theorem III.2.5(v), either $a = r_1 p$ or $b = r_2 p$; that is, either $p \mid a$ or $p \mid b$. Hence $p$ is prime.

# Theorem III.3.4 (continued 1)

**Theorem III.3.4.** Let $p$ and $c$ be nonzero elements in an integral domain $R$.

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Let $(p)$ be a nonzero prime ideal. Let $p \mid ab$. Then $ab = rp$ for some $r \in R$ and $ab \in (p)$. Since $R$ is commutative, Theorem III.2.15 implies that either $a \in (p)$ or $b \in (p)$. So, by Theorem III.2.5(v), either $a = r_1 p$ or $b = r_2 p$; that is, either $p \mid a$ or $p \mid b$. Hence $p$ is prime.

# Theorem III.3.4 (continued 2)

**Theorem III.3.4.** Let $p$ and $c$ be nonzero elements in an integral domain $R$.

(ii) $c$ is irreducible if and only if $(c)$ is maximal in the set $S$ of all proper principal ideals of $R$.

**Proof. (ii)** Suppose $c$ is irreducible. Then by definition, $c$ is a nonunit and so by Theorem III.3.2(iv), $(c)$ is a proper ideal of $R$ (i.e., $(c) \neq R$). If $(c) \subset (d)$ then by Theorem III.3.2(i), $c = dx$ for some $x \in R$. Since $c$ is irreducible then (by definition) either $d$ is a unit or $x$ is a unit. If $d$ is a unit then by Theorem III.3.2(iv), $(d) = R$. If $x$ is a unit then by Theorem III.3.2(vi), $c$ and $d$ are associates and then by Theorem III.3.2(ii), $(c) = (d)$. Hence $(c)$ is maximal in the set of proper principal ideals of $R$.

# Theorem III.3.4 (continued 3)

**Theorem III.3.4.** Let $p$ and $c$ be nonzero elements in an integral domain $R$.

(ii) $c$ is irreducible if and only if $(c)$ is maximal in the set $S$ of all proper principal ideals of $R$.

**Proof (continued). (ii)** Conversely, suppose $(c)$ is maximal in set $S$. Then $c$ is a nonzero nonunit in $R$ by Theorem III.3.2(iv) since $(c) \neq R$. If $c = ab$ then $(c) \subset (a)$ by Theorem III.3.2(i), whence $(c) = (a)$ or $(a) = R$. If $(a) = R$ then $a$ is a unit by Theorem III.3.2(iv). If $(c) = (a)$ then $a = cy$ (as in part (i) or by Theorem III.3.2(i)) and $c = ab = (cy)b = c(yb)$. Since $R$ is an integral domain then by left cancellation (which holds by Lemma III.1.A) $1_R = yb$ whence $b$ is a unit. Hence, either $a$ is a unit or $b$ is a unit, and $c$ is irreducible.

# Theorem III.3.4 (continued 4)

**Theorem III.3.4.** Let $p$ and $c$ be nonzero elements in an integral domain $R$.

(iii) Every prime element of $R$ is irreducible.

**Proof. (iii)** Let $p$ be prime in $R$. If $p = ab$ then either $p \mid a$ or $p \mid b$. WLOG, say $p \mid a$. Then $a = px$ for some $x \in R$ and so $p = ab = pxb$. Since $R$ is an integral domain (no zero divisors) then left cancellation holds (by Lemma III.1.A) and $1_R = xb$. Therefore $b$ is a unit. So (by definition) $p$ is irreducible.

# Theorem III.3.4 (continued 5)

**Theorem III.3.4.** Let $p$ and $c$ be nonzero elements in an integral domain $R$.

(iv) If $R$ is a principal ideal domain, then $p$ is prime if and only if $p$ is irreducible.

**Proof. (iv)** Suppose $R$ is a principal ideal domain. If $p$ is irreducible then by (ii), $(p)$ is maximal in the set of all proper principal ideals of $R$. Since $R$ is a principal ideal domain then (by definition) every ideal is principal, so in fact $(p)$ is maximal in $R$ itself. Since $R$ is an integral domain by hypothesis, then (by definition) $R$ has an identity. So by Theorem III.2.19, $(p)$ is prime. By (i), $p$ is prime (notice that $p \neq 0$ since the definition of "irreducible" implies $p \neq 0$).

Conversely, suppose $p$ is prime. Then by (iii) $p$ is irreducible (all we need for this is the hypothesis that $R$ is an integral domain).

# Theorem III.3.4 (continued 5)

**Theorem III.3.4.** Let $p$ and $c$ be nonzero elements in an integral domain $R$.

(iv) If $R$ is a principal ideal domain, then $p$ is prime if and only if $p$ is irreducible.

**Proof. (iv)** Suppose $R$ is a principal ideal domain. If $p$ is irreducible then by (ii), $(p)$ is maximal in the set of all proper principal ideals of $R$. Since $R$ is a principal ideal domain then (by definition) every ideal is principal, so in fact $(p)$ is maximal in $R$ itself. Since $R$ is an integral domain by hypothesis, then (by definition) $R$ has an identity. So by Theorem III.2.19, $(p)$ is prime. By (i), $p$ is prime (notice that $p \neq 0$ since the definition of "irreducible" implies $p \neq 0$).

Conversely, suppose $p$ is prime. Then by (iii) $p$ is irreducible (all we need for this is the hypothesis that $R$ is an integral domain).

# Theorem III.3.4 (continued 6)

**Theorem III.3.4.** Let $p$ and $c$ be nonzero elements in an integral domain $R$.

(v) Every associate of an irreducible (respectively, prime) element of $R$ is irreducible (respectively, prime).

**Proof. (v)** Let $c$ be irreducible and $d$ an associate of $c$. Then by Theorem III.3.2(vi) $c = du$ where $u \in R$ is a unit. To show $d$ is irreducible, suppose $d = ab$. Then $c = du = abu$, whence (since $c$ is irreducible) either $a$ or $bu$ is a unit. But if $bu$ is a unit then $(bu)v = 1$ for some $v \in R$ and so $b(uv) = 1$ and $b$ is a unit. So either $a$ or $b$ is a unit and $d$ is irreducible.

Let $c$ be prime and $d$ an associate of $c$. Then by Theorem III.3.2(vi), $c = du$ where $u \in R$ is a unit. To show $d$ is prime, suppose $d \mid ab$. Then $dx = ab$ for some $x \in R$. So $dux = abu$ or $cx = abu$. Since $c$ is prime, either $c$ divides $a$ or $c$ divides $bu$. If $c$ divides $a$ then $cy = a$ for some $y \in R$ and so $duy = (du)y = cy = a$ and $d$ divides $a$. If $c$ divides $bu$ then $cz = bu$ for some $z \in R$, or $(du)z = bu$ or $dz = b$ (since $u$ is a unit) and $d$ divides $b$. So $d$ is prime.

# Theorem III.3.4 (continued 6)

**Theorem III.3.4.** Let $p$ and $c$ be nonzero elements in an integral domain $R$.

(v) Every associate of an irreducible (respectively, prime) element of $R$ is irreducible (respectively, prime).

**Proof. (v)** Let $c$ be irreducible and $d$ an associate of $c$. Then by Theorem III.3.2(vi) $c = du$ where $u \in R$ is a unit. To show $d$ is irreducible, suppose $d = ab$. Then $c = du = abu$, whence (since $c$ is irreducible) either $a$ or $bu$ is a unit. But if $bu$ is a unit then $(bu)v = 1$ for some $v \in R$ and so $b(uv) = 1$ and $b$ is a unit. So either $a$ or $b$ is a unit and $d$ is irreducible. Let $c$ be prime and $d$ an associate of $c$. Then by Theorem III.3.2(vi), $c = du$ where $u \in R$ is a unit. To show $d$ is prime, suppose $d \mid ab$. Then $dx = ab$ for some $x \in R$. So $dux = abu$ or $cx = abu$. Since $c$ is prime, either $c$ divides $a$ or $c$ divides $bu$. If $c$ divides $a$ then $cy = a$ for some $y \in R$ and so $duy = (du)y = cy = a$ and $d$ divides $a$. If $c$ divides $bu$ then $cz = bu$ for some $z \in R$, or $(du)z = bu$ or $dz = b$ (since $u$ is a unit) and $d$ divides $b$. So $d$ is prime.

# Theorem III.3.4 (continued 7)

**Theorem III.3.4.** Let $p$ and $c$ be nonzero elements in an integral domain $R$.

(vi) The only divisors of an irreducible element of $R$ are its associates and the units of $R$.

**Proof. (vi)** Let $c$ be irreducible and suppose $a$ is a divisor of $c$: $a \mid c$. By Theorem III.3.2(i), $(c) \subset (a)$. By (ii), since $c$ is irreducible, then $(c)$ is maximal in the set of principal ideals of $R$. So it must be that either $(a) = (c)$ or $(a) = R$. If $(a) = (c)$ then $a$ is an associate of $c$ by Theorem III.3.2(ii). If $(a) = R$ then $a$ is a unit by Theorem III.3.2(iv). So $a$ is either an associate of $c$ or a unit. That is, the divisors of $c$ are its associates and the units of $R$, as claimed. □

# Lemma III.3.6

**Lemma III.3.6.** If $R$ is a principal ideal ring and $(a_1) \subset (a_2) \subset \cdots$ is a chain of ideals in $R$, then for some positive integer $n$, $(a_j) = (a_n)$ for all $j \geq n$.

**Proof.** Let $A = \cup_i (a_i)$. We claim that $A$ is an ideal. If $b, c \in A$, then $b \in (a_i)$ and $c \in (a_j)$ for some $i, j$. WLOG, say $i \geq j$. Consequently, $(a_j) \subset (a_i)$ and $b, c \in (a_i)$. Since $(a_i)$ is an ideal then by Theorem III.2.2(i) $b - c \in (a_i) \subset A$. Similarly if $r \in R$ and $b \in A$ then $b \in (a_i)$ for some $i$, whence $rb \in (a_i) \subset A$ and $br \in (a_i) \subset A$ by Theorem III.2.2(ii). Then, Theorem III.2.2 implies that $A$ is an ideal.

# Lemma III.3.6

**Lemma III.3.6.** If $R$ is a principal ideal ring and $(a_1) \subset (a_2) \subset \cdots$ is a chain of ideals in $R$, then for some positive integer $n$, $(a_j) = (a_n)$ for all $j \geq n$.

**Proof.** Let $A = \cup_i (a_i)$. We claim that $A$ is an ideal. If $b, c \in A$, then $b \in (a_i)$ and $c \in (a_j)$ for some $i, j$. WLOG, say $i \geq j$. Consequently, $(a_j) \subset (a_i)$ and $b, c \in (a_i)$. Since $(a_i)$ is an ideal then by Theorem III.2.2(i) $b - c \in (a_i) \subset A$. Similarly if $r \in R$ and $b \in A$ then $b \in (a_i)$ for some $i$, whence $rb \in (a_i) \subset A$ and $br \in (a_i) \subset A$ by Theorem III.2.2(ii). Then, Theorem III.2.2 implies that $A$ is an ideal. Since $R$ is a principal ideal ring (by definition, all ideals are principal) then $A$ is principal, say $A = (a)$. Since $a \in A = \cup(a_i)$ then $a \in (a_n)$ for some $n$. By Definition III.2.4 (of a generating set of an ideal), $A = (a) \subset (a_n)$. Therefore, for every $j \geq n$ we have $A = (a) \subset (a_n) \subset (a_j) \subset A$ and whence $(a_j) = (a_n)$. □

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# Theorem III.3.7

**Theorem III.3.7.** Every principal ideal domain $R$ is a unique factorization domain. That is, "every PID is a UFD."

**Proof.** Let $S$ be the set of all nonzero nonunit elements of $R$ which cannot be factored as a finite product of irreducible elements. We first show that $S$ is empty (and so all nonzero nonunits have at least one factorization into irreducibles).

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ASSUME $S$ is not empty and $a \in S$. Then $a$ is not a unit and by Theorem III.3.2(iv), $(a) \neq R$ is a proper ideal of $R$. By Theorem III.2.18, $(a) \subset (c)$ where $(c)$ is some maximal ideal (since $R$ is a principal ideal domain, by definition, all ideals are principal). By Theorem III.3.4(ii), $c$ is irreducible. Since $(a) \subset (c)$ then by Theorem III.3.2(i), $c$ divides $a$. So there is an irreducible $c$ dividing $a$ for any $a \in S$.

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# Theorem III.3.7 (continued 1)

**Theorem III.3.7.** Every principal ideal domain $R$ is a unique factorization domain. That is, "every PID is a UFD."

**Proof (continued).** We claim $x_a \in S$. First, ASSUME that $x_a$ is a unit (in which case $x_a \notin S$ by the definition of $S$), then $a = c_a x_a$ implies by Theorem III.3.2(iv) that $a$ and $c_a$ are associates. Then by Theorem III.3.4(v), $a$ is irreducible since $c_a$ is irreducible. But then $a$ is a finite product of irreducibles (namely, $a$ itself) and so $a \notin S$, a CONTRADICTION. So $x_a$ is not a unit. Let $x_a$ be a nonunit and ASSUME $x_a \notin S$. Then $x_a$ has a factorization as a product of irreducibles, whence $a$ also does (just add irreducible $c_a$ to the product), which implies that $a \notin S$, a CONTRADICTION. So it must be that $x_a \in S$.

# Theorem III.3.7 (continued 2)

**Proof (continued).** Furthermore, we claim that $(a)$ is properly contained in $(x_a)$. Since $x_a \mid a$ then by Theorem III.3.2(i) we have $(a) \subset (x_a)$. ASSUME $(a) = (x_a)$. Then by Theorem III.3.2(ii), $a$ and $x_a$ are associates and by Theorem III.3.2(vi) (since $R$ is an integral domain) $x_a = ay$ for some unit $y \in R$, whence $a = x_a c_a = ayc_a$ and $1 = yc_a$ (cancellation holds in an integral domain by Lemma III.1.A). But then $c_a$ is a unit, a CONTRADICTION since $c_a$ is irreducible (and by definition, "irreducibles" are nonunits). Therefore, $(a) \subsetneq (x_a)$.

We now define $f : S \to S$ as $f(a) = x_a$ where $x_a$ is described above (and $f$ is well-defined since $x_a$ is uniquely determined by $c_a$; but the Axiom of Choice is required to get $c_a$). For each $n \in \mathbb{N} \cup \{0\}$, define $f_n = f$. Then by the Recursion Theorem (Theorem 0.6.2) there exists a function $\varphi : \mathbb{N} \cup \{0\} \to S$ such that $\varphi(0) = a$ and $\varphi(n+1) = f_n(\varphi(n)) = f(\varphi(n)) = x_{\varphi(n)}$ (this last one from the definition of $f$) for all $n \in \mathbb{N} \cup \{0\}$.

# Theorem III.3.7 (continued 2)

**Proof (continued).** Furthermore, we claim that $(a)$ is properly contained in $(x_a)$. Since $x_a \mid a$ then by Theorem III.3.2(i) we have $(a) \subset (x_a)$. ASSUME $(a) = (x_a)$. Then by Theorem III.3.2(ii), $a$ and $x_a$ are associates and by Theorem III.3.2(vi) (since $R$ is an integral domain) $x_a = ay$ for some unit $y \in R$, whence $a = x_a c_a = ayc_a$ and $1 = yc_a$ (cancellation holds in an integral domain by Lemma III.1.A). But then $c_a$ is a unit, a CONTRADICTION since $c_a$ is irreducible (and by definition, "irreducibles" are nonunits). Therefore, $(a) \subsetneq (x_a)$.
We now define $f : S \to S$ as $f(a) = x_a$ where $x_a$ is described above (and $f$ is well-defined since $x_a$ is uniquely determined by $c_a$; but the Axiom of Choice is required to get $c_a$). For each $n \in \mathbb{N} \cup \{0\}$, define $f_n = f$. Then by the Recursion Theorem (Theorem 0.6.2) there exists a function $\varphi : \mathbb{N} \cup \{0\} \to S$ such that $\varphi(0) = a$ and $\varphi(n+1) = f_n(\varphi(n)) = f(\varphi(n)) = x_{\varphi(n)}$ (this last one from the definition of $f$) for all $n \in \mathbb{N} \cup \{0\}$.

# Theorem III.3.7 (continued 3)

**Proof (continued).** Denote $\varphi(n) \in S$ as $a_n$ and then we have the sequence $a_0 = a, a_1, a_2, \ldots$ such that (since $\varphi(n+1) = x_{\varphi(n)}$): $a_1 = x_a; a_2 = x_{a_1}; a_3 = x_{a_2}; \cdots; a_{n+1} = x_{a_n}; \cdots$. Consequently, since from the previous paragraph $(a) \subsetneq (x_a)$, we have $(a) \subsetneq (a_1) \subsetneq (a_2) \subsetneq (a_3) \subsetneq \cdots$, but this CONTRADICTS Lemma III.3.6, so the assumption that $S$ is not empty is false, and so $S = \varnothing$. That is, every nonzero nonunit in $R$ has a factorization as a finite product of irreducibles and (i) of the definition on UFD follows.

Finally, if $c_1 c_2 \cdots c_n = d_1 d_2 \cdots d_m$ are two factorizations of an element of $R$ into a product of irreducibles, then $c_1$ is prime by Theorem III.3.4(iv) and so (by definition of prime, and some induction) $c_1$ divides some $d_i$. Since $d_i$ is irreducible, then by Theorem III.3.4(vi) $c_1$ and $d_i$ are associates (since $c_1$ is irreducible, it is not a unit).

# Theorem III.3.7 (continued 3)

**Proof (continued).** Denote $\varphi(n) \in S$ as $a_n$ and then we have the sequence $a_0 = a, a_1, a_2, \ldots$ such that (since $\varphi(n+1) = x_{\varphi(n)}$): $a_1 = x_a; a_2 = x_{a_1}; a_3 = x_{a_2}; \cdots; a_{n+1} = x_{a_n}; \cdots$. Consequently, since from the previous paragraph $(a) \subsetneq (x_a)$, we have $(a) \subsetneq (a_1) \subsetneq (a_2) \subsetneq (a_3) \subsetneq \cdots$, but this CONTRADICTS Lemma III.3.6, so the assumption that $S$ is not empty is false, and so $S = \varnothing$. That is, every nonzero nonunit in $R$ has a factorization as a finite product of irreducibles and (i) of the definition on UFD follows.

Finally, if $c_1 c_2 \cdots c_n = d_1 d_2 \cdots d_m$ are two factorizations of an element of $R$ into a product of irreducibles, then $c_1$ is prime by Theorem III.3.4(iv) and so (by definition of prime, and some induction) $c_1$ divides some $d_i$. Since $d_i$ is irreducible, then by Theorem III.3.4(vi) $c_1$ and $d_i$ are associates (since $c_1$ is irreducible, it is not a unit).

# Theorem III.3.7 (continued 4)

**Theorem III.3.7.** Every principal ideal domain $R$ is a unique factorization domain. That is, "every PID is a UFD."

**Proof (continued).** By Theorem III.3.2(vi), we then have that $c_1 = d_i r_1$ for some unit $r_1 \in R$ ($R$ is given to be an integral domain). So $c_1 c_2 \cdots c_n = d_1 d_2 \cdots d_m$ implies that $(d_i r_1) c_2 c_3 \cdots c_n = d_1 d_2 \cdots d_i \cdots d_m$ or $r_1 c_2 c_3 \cdots c_n = d_1 d_2 \cdots d_{i-1} d_{i+1} \cdots d_m$. (WLOG $n \leq m$.) Then we similarly have that $r_1 r_2 \cdots r_m = \underbrace{d_\alpha d_\beta \ldots d_\omega}_{m-n \text{ factors}}$.

ASSUME $m - n > 0$. Then we can reverse the argument above and use the facts that $d_\alpha$ is prime (Theorem III.3.4(iv)) and $d_\alpha$ divides $r_1 r_2 \cdots r_m$ to see that $d_\alpha$ must divide some unit $r_j$. But then $d_\alpha x = r_j$ for some $x \in R$. But since $r_j$ is a unit, then $d_\alpha (x r_j^{-1}) = 1$ (integral domains have 1) and so $d_\alpha$ is a unit, CONTRADICTING the fact that $d_\alpha$ is irreducible and hence is not a unit (by the definition of "irreducible"). So $m - n = 0$, $m = n$, and (ii) of the definition of UFD holds. □

# Theorem III.3.7 (continued 4)

**Theorem III.3.7.** Every principal ideal domain $R$ is a unique factorization domain. That is, "every PID is a UFD."

**Proof (continued).** By Theorem III.3.2(vi), we then have that $c_1 = d_i r_1$ for some unit $r_1 \in R$ ($R$ is given to be an integral domain). So $c_1 c_2 \cdots c_n = d_1 d_2 \cdots d_m$ implies that $(d_i r_1) c_2 c_3 \cdots c_n = d_1 d_2 \cdots d_i \cdots d_m$ or $r_1 c_2 c_3 \cdots c_n = d_1 d_2 \cdots d_{i-1} d_{i+1} \cdots d_m$. (WLOG $n \leq m$.) Then we similarly have that $r_1 r_2 \cdots r_m = \underbrace{d_\alpha d_\beta \ldots d_\omega}_{m-n \text{ factors}}$.

ASSUME $m - n > 0$. Then we can reverse the argument above and use the facts that $d_\alpha$ is prime (Theorem III.3.4(iv)) and $d_\alpha$ divides $r_1 r_2 \cdots r_m$ to see that $d_\alpha$ must divide some unit $r_j$. But then $d_\alpha x = r_j$ for some $x \in R$. But since $r_j$ is a unit, then $d_\alpha (x r_j^{-1}) = 1$ (integral domains have 1) and so $d_\alpha$ is a unit, CONTRADICTING the fact that $d_\alpha$ is irreducible and hence is not a unit (by the definition of "irreducible"). So $m - n = 0$, $m = n$, and (ii) of the definition of UFD holds. □

# Theorem III.3.9

**Theorem III.3.9.** Every Euclidean ring $R$ is a principal ideal ring with identity. Consequently every Euclidean domain is a unique factorization domain.

**Proof.** If $I$ is a nonzero ideal in $R$, choose $a \in I$ such that $\varphi(a)$ is the least nonnegative integer in the set $\{\varphi(x) \mid x \neq 0; x \in I\}$ (where $\varphi : R \setminus \{0\} \to \mathbb{N}$ as described in Definition III.3.8; so $a$ is a "$\varphi$-least nonzero element of $I$"). If $b \in I$ then $b = qa + r$ for some $q, r \in R$ with either $r = 0$ or $r \neq 0$ and $\varphi(r) < \varphi(a)$ (by part (ii) of Definition III.3.8). Since $b \in I$ and $qa \in I$ ($I$ is an ideal), then $r = b - qa \in I$. Since $\varphi(a)$ is minimal as chosen, then we cannot have $\varphi(r) < \varphi(a)$ if $r \neq 0$, and so it must be that $r = 0$ (that is, $r \in I$ but $r \notin \{x \in I \mid x \neq 0\}$; the $x$'s for which we consider $\varphi(x)$ above). Whence $b = qa$ and $a$ divides $b$. Therefore, every element of $I$ is a multiple of $a$. So $I \subset Ra$.

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# Theorem III.3.9 (continued)

**Proof (continued).** Since Euclidean ring $R$ is a commutative ring, $a$ is in the center of $R$ and so by Theorem III.2.5(iii), $Ra \subset (a)$. Since $a \in I$, then $(a) \subset I$. So we have $I \subset Ra \subset (a) \subset I$. Therefore $I = Ra = (a)$ and arbitrary ideal $I$ of $R$ is principal. So $R$ is a principal ideal ring.

In the previous paragraph, we showed that any nonzero ideal $I$ satisfies $I = Ra = (a)$. Since $R$ itself is a nonzero ideal of $R$, then we have $R = Ra = (a)$ for some $a \in R$. Consequently, for some $e \in R$ we have $a = ea = ae$. If $b \in R = Ra$ then $b = xa$ for some $x \in R$. Therefore, $be = (xa)e = x(ae) = xa = b$, whence $e$ is a multiplicative identity element for $R$.

# Theorem III.3.9 (continued)

**Proof (continued).** Since Euclidean ring $R$ is a commutative ring, $a$ is in the center of $R$ and so by Theorem III.2.5(iii), $Ra \subset (a)$. Since $a \in I$, then $(a) \subset I$. So we have $I \subset Ra \subset (a) \subset I$. Therefore $I = Ra = (a)$ and arbitrary ideal $I$ of $R$ is principal. So $R$ is a principal ideal ring.

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In the first paragraph we have that a Euclidean ring is a principal idea ring. By Theorem III.3.7, every principal ideal domain is a unique factorization domain. "Whence" every Euclidean domain is a unique factorization domain. □

# Theorem III.3.9 (continued)

**Proof (continued).** Since Euclidean ring $R$ is a commutative ring, $a$ is in the center of $R$ and so by Theorem III.2.5(iii), $Ra \subset (a)$. Since $a \in I$, then $(a) \subset I$. So we have $I \subset Ra \subset (a) \subset I$. Therefore $I = Ra = (a)$ and arbitrary ideal $I$ of $R$ is principal. So $R$ is a principal ideal ring.

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# Theorem III.3.11

**Theorem III.3.11.** Let $a_1, a_2, \ldots, a_n$ be elements of a commutative ring $R$ with identity.

(i) $d \in R$ is a greatest common divisor of $\{a_1, a_2, \ldots, a_n\}$ such that $d = r_1a_1 + r_2a_2 + \cdots + r_na_n$ for some $r_i \in R$ if and only if $(d) = (a_1) + (a_2) + \cdots + (a_n)$.

(ii) If $R$ is a principal ideal ring, then a greatest common divisor of $a_1, a_2, \ldots, a_n$ exists and every one is of the form $r_1a_1 + r_2a_2 + \cdots r_na_n$, where each $r_i \in R$.

(iii) If $R$ is a unique factorization domain, then there exists a greatest common divisor of $a_1, a_2, \ldots, a_n$.

**Proof.** **(i)** Suppose $d = r_1a_1 + r_2a_2 + \cdots + r_na_n$ is a greatest common divisor of $\{a_1, a_2, \ldots, a_n\}$. Then by Definition III.3.10(i), $d \mid a_1, d \mid a_2, \ldots, d \mid a_n$ and so by Theorem III.3.2(i), $(a_1) \subset (d), (a_2) \subset (d), \ldots, (a_n) \subset (d)$.

# Theorem III.3.11

**Theorem III.3.11.** Let $a_1, a_2, \ldots, a_n$ be elements of a commutative ring $R$ with identity.

(i) $d \in R$ is a greatest common divisor of $\{a_1, a_2, \ldots, a_n\}$ such that $d = r_1a_1 + r_2a_2 + \cdots + r_na_n$ for some $r_i \in R$ if and only if $(d) = (a_1) + (a_2) + \cdots + (a_n)$.

(ii) If $R$ is a principal ideal ring, then a greatest common divisor of $a_1, a_2, \ldots, a_n$ exists and every one is of the form $r_1a_1 + r_2a_2 + \cdots r_na_n$, where each $r_i \in R$.

(iii) If $R$ is a unique factorization domain, then there exists a greatest common divisor of $a_1, a_2, \ldots, a_n$.

**Proof. (i)** Suppose $d = r_1a_1 + r_2a_2 + \cdots + r_na_n$ is a greatest common divisor of $\{a_1, a_2, \ldots, a_n\}$. Then by Definition III.3.10(i), $d \mid a_1, d \mid a_2, \ldots, d \mid a_n$ and so by Theorem III.3.2(i), $(a_1) \subset (d), (a_2) \subset (d), \ldots, (a_n) \subset (d)$.

# Theorem III.3.11 (continued 1)

**Theorem III.3.11.** Let $a_1, a_2, \ldots, a_n$ be elements of a commutative ring $R$ with identity.

(i) $d \in R$ is a greatest common divisor of $\{a_1, a_2, \ldots, a_n\}$ such that $d = r_1a_1 + r_2a_2 + \cdots + r_na_n$ for some $r_i \in R$ if and only if $(d) = (a_1) + (a_2) + \cdots + (a_n)$.

**Proof (continued). (i)** Since $(d)$ is an ideal it is closed under addition and so $(a_1) + (a_2) + \cdots + (a_n) \subset (d)$. But $d = r_1a_1 + r_2a_2 + \cdots r_na_n \in (a_1) + (a_2) + \cdots + (a_n)$. So $(a_1) + (a_2) + \cdots + (a_n)$ is an ideal (Theorem III.2.6(i)) containing $d$ and so $(d) \subset (a_1) + (a_2) + \cdots + (a_n)$. Therefore $(d) = (a_1) + (a_2) + \cdots + (a_n)$.

Now suppose $(d) = (a_1) + (a_2) + \cdots + (a_n)$. Since $R$ has an identity and $a_1, a_2, \ldots, a_n$ are in the center of $R$ (since $R$ is commutative), by Theorem III.2.5(v) we have that $(a_i) = a_iR = Ra_i$ for each $a_i$. Also, $d \in (d)$ by Theorem III.2.5(iv), so $d = r_1a_1 + r_2a_2 + \cdots + r_na_n$ for some $r_i \in R$. Now $(d) = (a_1) + (a_2) + \cdots + (a_n)$ implies that each $(a_i) \subset (d)$ and so by Theorem III.3.2(i), $d \mid a_i$ for each $a_i$. So $d$ is a divisor of each $a_i$.

# Theorem III.3.11 (continued 1)

**Theorem III.3.11.** Let $a_1, a_2, \ldots, a_n$ be elements of a commutative ring $R$ with identity.

(i) $d \in R$ is a greatest common divisor of $\{a_1, a_2, \ldots, a_n\}$ such that $d = r_1a_1 + r_2a_2 + \cdots + r_na_n$ for some $r_i \in R$ if and only if $(d) = (a_1) + (a_2) + \cdots + (a_n)$.

**Proof (continued). (i)** Since $(d)$ is an ideal it is closed under addition and so $(a_1) + (a_2) + \cdots + (a_n) \subset (d)$. But $d = r_1a_1 + r_2a_2 + \cdots r_na_n \in (a_1) + (a_2) + \cdots + (a_n)$. So $(a_1) + (a_2) + \cdots + (a_n)$ is an ideal (Theorem III.2.6(i)) containing $d$ and so $(d) \subset (a_1) + (a_2) + \cdots + (a_n)$. Therefore $(d) = (a_1) + (a_2) + \cdots + (a_n)$. Now suppose $(d) = (a_1) + (a_2) + \cdots + (a_n)$. Since $R$ has an identity and $a_1, a_2, \ldots, a_n$ are in the center of $R$ (since $R$ is commutative), by Theorem III.2.5(v) we have that $(a_i) = a_iR = Ra_i$ for each $a_i$. Also, $d \in (d)$ by Theorem III.2.5(iv), so $d = r_1a_1 + r_2a_2 + \cdots + r_na_n$ for some $r_i \in R$. Now $(d) = (a_1) + (a_2) + \cdots + (a_n)$ implies that each $(a_i) \subset (d)$ and so by Theorem III.3.2(i), $d \mid a_i$ for each $a_i$. So $d$ is a divisor of each $a_i$.

# Theorem III.3.11 (continued 2)

**Theorem III.3.11.** Let $a_1, a_2, \ldots, a_n$ be elements of a commutative ring $R$ with identity.

(i) $d \in R$ is a greatest common divisor of $\{a_1, a_2, \ldots, a_n\}$ such that $d = r_1a_1 + r_2a_2 + \cdots + r_na_n$ for some $r_i \in R$ if and only if $(d) = (a_1) + (a_2) + \cdots + (a_n)$.

**Proof (continued). (i)** Suppose now that $c$ also divides each $a_i$: $c \mid a_i$ for each $a_i$. Then Theorem III.3.2(i) implies that $(a_i) \subset (c)$ for each $a_i$. But then $(d) = (a_1) + (a_2) + \cdots + (a_n) \subset (c)$ (since $(c)$ is an ideal and hence closed under addition) and again by Theorem III.3.2(i) we have that $c \mid d$. Therefore, by Definition III.3.10, $d$ is the greatest common divisor of $a_1, a_2, \ldots, a_n$.

# Theorem III.3.11 (continued 3)

**Theorem III.3.11.** Let $a_1, a_2, \ldots, a_n$ be elements of a commutative ring $R$ with identity.

(ii) If $R$ is a principal ideal ring, then a greatest common divisor of $a_1, a_2, \ldots, a_n$ exists and every one is of the form $r_1a_1 + r_2a_2 + \cdots r_na_n$, where each $r_i \in R$.

**Proof. (ii)** Suppose $R$ is a principal ideal ring and let $a_1, a_2, \ldots, a_n \in R$. By Theorem III.2.6(i), $(a_1) + (a_2) + \cdots + (a_n)$ is an ideal of $R$. Since $R$ is a principal ideal ring, then $(a_1) + (a_2) + \cdots + (a_n) = (d)$ for some $d \in R$. By (i), we have that $d$ is a greatest common divisor of $a_1, a_2, \ldots, a_n$ and is of the form $d = r_1a_1 + r_2a_2 + \cdots + r_na_n$.

# Theorem III.3.11 (continued 4)

**Theorem III.3.11.** Let $a_1, a_2, \ldots, a_n$ be elements of a commutative ring $R$ with identity.

(iii) If $R$ is a unique factorization domain, then there exists a greatest common divisor of $a_1, a_2, \ldots, a_n$.

**Proof. (iii)** Suppose $R$ is a unique factorization domain. Then for each $a_i$ we have $a_i = c_1^{m_{i1}} c_2^{m_{i2}} \cdots c_t^{m_{it}}$ for distinct irreducible $c_k$'s and with each $m_{ij} \geq 0$ (we get the $c_k$'s for each $a_i$ and then use exponents of 0 as necessary). Then $d = c_1^{k_1} c_2^{k_2} \cdots c_t^{k_t}$ where $k_j = \min\{m_{1j}, m_{2j}, \ldots, m_{nj}\}$ is a divisor of each $a_1, a_2, \ldots, a_n$.

Let $c$ be a divisor of each $a_i$. Write $c$ in terms of irreducibles $d_i$: $c = d_1 d_2 \cdots d_\ell$, which can be done since $R$ is a unique factorization domain. Since $c = d_1 d_2 \cdots d_\ell$ divides $a_i = c_1^{m_{i1}} c_2^{m_{i2}} \cdots c_t^{m_{it}}$, then each $d_j$ divides $a_i$. Now each of the $c_i$'s are irreducible (and so are not units by definition), so $d_j \mid a_i$ implies that irreducible $d_j = c_i$ for some $c_i$.

# Theorem III.3.11 (continued 4)

**Theorem III.3.11.** Let $a_1, a_2, \ldots, a_n$ be elements of a commutative ring $R$ with identity.

(iii) If $R$ is a unique factorization domain, then there exists a greatest common divisor of $a_1, a_2, \ldots, a_n$.

**Proof. (iii)** Suppose $R$ is a unique factorization domain. Then for each $a_i$ we have $a_i = c_1^{m_{i1}} c_2^{m_{i2}} \cdots c_t^{m_{it}}$ for distinct irreducible $c_k$'s and with each $m_{ij} \geq 0$ (we get the $c_k$'s for each $a_i$ and then use exponents of 0 as necessary). Then $d = c_1^{k_1} c_2^{k_2} \cdots c_t^{k_t}$ where $k_j = \min\{m_{1j}, m_{2j}, \ldots, m_{nj}\}$ is a divisor of each $a_1, a_2, \ldots, a_n$.

Let $c$ be a divisor of each $a_i$. Write $c$ in terms of irreducibles $d_i$: $c = d_1 d_2 \cdots d_\ell$, which can be done since $R$ is a unique factorization domain. Since $c = d_1 d_2 \cdots d_\ell$ divides $a_i = c_1^{m_{i1}} c_2^{m_{i2}} \cdots c_t^{m_{it}}$, then each $d_j$ divides $a_i$. Now each of the $c_i$'s are irreducible (and so are not units by definition), so $d_j \mid a_i$ implies that irreducible $d_j = c_i$ for some $c_i$.

# Theorem III.3.11 (continued 5)

**Theorem III.3.11.** Let $a_1, a_2, \ldots, a_n$ be elements of a commutative ring $R$ with identity.

(iii) If $R$ is a unique factorization domain, then there exists a greatest common divisor of $a_1, a_2, \ldots, a_n$.

**Proof (continued). (iii)** Hence $c = c_1^{n_1} c_2^{n_2} \cdots c_t^{n_t}$ for some $n_i \geq 0$. If each $n_i \leq k_i$ then $c \mid d$. ASSUME $n_{i^*} > k_{i^*}$ for some $i^*$. Then there is an $a_i$ where $a_i = c_1^{m_{i1}} c_2^{m_{i2}} \cdots c_{i^*}^{k_{i^*}} \cdots c_t^{m_{it}}$ and $c$ does not divide this $a_i$, a CONTRADICTION. So each $n_i \leq k_i$ and $c \mid d$. That is, $d$ is a greatest common divisor. □