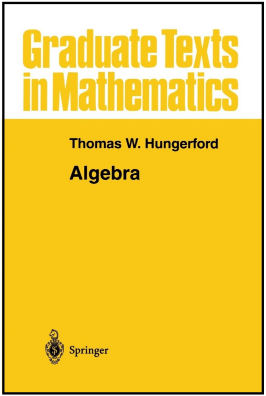
# Modern Algebra

## Chapter IV. Modules

IV.1. Modules, Homomorphisms, and Exact Sequences—Proofs of Theorems



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$$r(a+B) = rB \text{ for all } r \in R, a \in A.$$

The map $\pi : A \to A/B$ given by $a \mapsto a + B$ is an $R$-module epimorphism with kernel $B$.

**Proof.** By the definition of module, $A$ is an additive abelian group, so $B < A$ is a normal subgroup and $A \setminus B$ is defined and itself abelian. If $a + B = a' + B$ (as cosets of $B$) then $a - a' \in B$.

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Now to check $\pi$. Consider $\pi(a+b) = (a+b) + B - (a+B) + (b+B) = \pi(a) + \pi(b)$, and $\pi(ra) = (ra + B = r(a+B) = r\pi(a)$, so $\pi$ is a homomorphism. Since $B$ is the identity in $A \setminus B$ an d$b + B = B$ if and only if $b \in B$, then $\mathrm{Ker}(\pi) = B$. Finally, $\pi$ is clearly onto, so that $\pi$ is an epimorphism. □

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# Theorem IV.1.12

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**Proof.** By Theorem I.8.2 there is a unique *group* homomorphism $\varphi : C \to \prod A_i$ which has the desired property and $\varphi$ is given by (as seen in the proof of Theorem I.8.2) $\varphi(c) = \{\varphi_i(c)\}_{i \in I}$.

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# Theorem IV.1.13

**Theorem IV.1.13.** If $R$ is a ring, $\{A_i \mid i \in I\}$ a family of $R$-modules, $D$ an $R$-module, and $\{\psi_i : A_i \to D \mid i \in I\}$ a family of $R$-module homomorphisms, then there is a unique $R$-module homomorphism $\psi : \sum_{i \in I} A_i \to D$ such that $\psi \iota_i = \psi_i$ for all $i \in I$. $\sum_{i \in I} A_i$ is uniquely determined up to isomorphism by this property. In other words, $\sum_{i \in I} A_i$ is a coproduct in the category of $R$-modules.

**Proof.** By Theorem I.8.5 there is a unique abelian group homomorphism $\psi : \sum A_i \to D$ with the desired property and $\psi$ is given by (as seen in the proof of Theorem I.8.5) $\psi(\{a_i\}) = \sum_i \psi_i(a_i)$, where the sum is taken over the finite set of indices $i$ such that $q_i \neq 0$.

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$$\psi(c\{a_i\}) = \psi(\{ca_i\}) = \sum_i \psi_i(ca_i) = \sum_i c\psi_i(a_i) = s \sum_i \psi_i(a_i) = s\psi(\{a_i\})$$

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and $\psi$ is an $R$-module homomorphism. By Definition I.7.4 (with $S = \sum_i A_i$, $B = D$, $\psi_i = \psi_i$, $\iota_i = \iota_i$, and $\psi = \psi$), $\sum_i A_i$ is a coproduct in the category of $R$-modules. By Theorem I.7.5, $\sum_i A_i$ is uniquely determined up to isomorphism (or "equivalence"). □

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# Theorem IV.1.14

**Theorem IV.1.14.** Let $R$ be a ring and $A, A_1, A_2, \ldots, A_n$ $R$-modules. Then $A \cong A_i \oplus A_2 \oplus \cdots \oplus A_n$ if and only if for each $i = 1, 2, \ldots, n$ there are $R$-module homomorphisms $\pi_i : A \to A_i$ and $\iota_i : A_i \to A$ such that

(i) $\pi_i \iota_i = 1_{A_i}$ for $i = 1, 2, \ldots, n$;

(ii) $\pi_j \iota_i = 0$ for $i \neq j$;

(iii) $\iota_1 \pi_1 + \iota_2 \pi_2 + \cdots + \iota_n \pi_n = 1_A$.

**Proof.** First, suppose $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$. We take $\pi_i$ as the canonical projection and $\iota_i$ the canonical injection.

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# Theorem IV.1.14 (continued 1)

**Proof (continued).** Also, for $(a_1, a_2, \ldots, a_n) \in A$ we have

$$(\iota_1\pi_2 + \iota_2\pi_2 + \cdots + \iota_n\pi_n)(a_1, a_2, \ldots, a_n)$$

$$\begin{aligned} &= \iota_1\pi_1(a_1, a_2, \ldots, a_n) + \iota_2\pi_2(a_1, a_2, \ldots, a_n) + \cdots + \iota_n\pi_n(a_1, a_2, \ldots, a_n) \\ &= \iota_1(a_1) + \iota_2(a_2) + \cdots + \iota_n(a_n) \\ &= (a_1, e_2, e_2, \ldots, e_n) + (e_1, a_2, e_3, \ldots, e_n) + \cdots + (e_1, e_2, \ldots, a_n) \\ &= (a_1, a_2, \ldots, a_n) \end{aligned}$$

and $\iota_1\pi_1 + \iota_2\pi_2 + \cdots + \iota_n\pi_n = 1_A$ and (iii) holds. Similarly, if the isomorphism is $f : A \to A_1 \oplus A_2 \oplus \cdots \oplus A_n$, then replacing $\pi_i$ with $\pi_i f : A \to A_i$ and $\iota_i$ with $f^{-1}\iota_i : A_i \to A$ then (i) and (ii) still hold and $f^{-1}(\iota_1\pi_1 + \iota_2\pi_2 + \cdots + \iota_n\pi_n)f = 1_A$ implies (iii).

# Theorem IV.1.14 (continued 1)

**Proof (continued).** Also, for $(a_1, a_2, \ldots, a_n) \in A$ we have

$$(\iota_1\pi_2 + \iota_2\pi_2 + \cdots + \iota_n\pi_n)(a_1, a_2, \ldots, a_n)$$

$$\begin{aligned}
&= \iota_1\pi_1(a_1, a_2, \ldots, a_n) + \iota_2\pi_2(a_1, a_2, \ldots, a_n) + \cdots + \iota_n\pi_n(a_1, a_2, \ldots, a_n) \\
&= \iota_1(a_1) + \iota_2(a_2) + \cdots + \iota_n(a_n) \\
&= (a_1, e_2, e_2, \ldots, e_n) + (e_1, a_2, e_3, \ldots, e_n) + \cdots + (e_1, e_2, \ldots, a_n) \\
&= (a_1, a_2, \ldots, a_n)
\end{aligned}$$

and $\iota_1\pi_1 + \iota_2\pi_2 + \cdots + \iota_n\pi_n = 1_A$ and (iii) holds. Similarly, if the isomorphism is $f : A \to A_1 \oplus A_2 \oplus \cdots \oplus A_n$, then replacing $\pi_i$ with $\pi_i f : A \to A_i$ and $\iota_i$ with $f^{-1}\iota_i : A_i \to A$ then (i) and (ii) still hold and $f^{-1}(\iota_1\pi_1 + \iota_2\pi_2 + \cdots + \iota_n\pi_n)f = 1_A$ implies (iii).

# Theorem IV.1.14 (continued 2)

**Proof (continued).** Second, suppose that $R$-module homomorphism $\pi_i : A \to A_i$ and $\iota_i : A_i \to A$ satisfy (i), (ii), (iii). Let the canonical projections $\pi_i' : A_1 \oplus A_2 \oplus \cdots \oplus A_n \to A_i$ and the canonical injections $\iota_i' : A_i \to A_1 \oplus A_2 \oplus \cdots \oplus A_n$ as $\varphi = \iota_1' \pi_1 + \iota_2' \pi_2 + \cdots + \iota_n' \pi_n$.

# Theorem IV.1.14 (continued 2)

**Proof (continued).** Second, suppose that $R$-module homomorphism $\pi_i : A \to A_i$ and $\iota_i : A_i \to A$ satisfy (i), (ii), (iii). Let the canonical projections $\pi_i' : A_1 \oplus A_2 \oplus \cdots \oplus A_n \to A_i$ and the canonical injections $\iota_i' : A_i \to A_1 \oplus A_2 \oplus \cdots \oplus A_n$ as $\varphi = \iota_1'\pi_1 + \iota_2'\pi_2 + \cdots + \iota_n'\pi_n$. Then

$$\begin{aligned}
\varphi\psi &= \sum_{i=1}^{n} \iota_i\pi_i' \sum_{j=1}^{n} \iota_j'\pi_j = \sum_{i=1}^{n}\sum_{j=1}^{n} \iota_i\pi_i'\iota_j'\pi_j \\
&= \sum_{i=1}^{n} \iota_i\pi_i'\iota_i'\pi_i \text{ since the canonical mappings satisfy} \\
&\qquad\qquad \pi_i\iota_j = 0 \text{ for } i \neq j \text{ as shown above} \\
&= \sum_{i=1}^{n} \iota_i 1_{A_i}\pi_i \text{ since } \pi_i'\iota_i' = 1_{A_i} \text{ by above} \\
&= \sum_{i=1}^{n} \iota_i\pi_i = 1_{A_i} \text{ by (iii) above.}
\end{aligned}$$

# Theorem IV.1.14 (continued 2)

**Proof (continued).** Second, suppose that $R$-module homomorphism $\pi_i : A \to A_i$ and $\iota_i : A_i \to A$ satisfy (i), (ii), (iii). Let the canonical projections $\pi_i' : A_1 \oplus A_2 \oplus \cdots \oplus A_n \to A_i$ and the canonical injections $\iota_i' : A_i \to A_1 \oplus A_2 \oplus \cdots \oplus A_n$ as $\varphi = \iota_1'\pi_1 + \iota_2'\pi_2 + \cdots + \iota_n'\pi_n$. Then

$$\begin{aligned}
\varphi\psi &= \sum_{i=1}^n \iota_i\pi_i' \sum_{j=1}^n \iota_j'\pi_j = \sum_{i=1}^n \sum_{j=1}^n \iota_i\pi_i'\iota_j'\pi_j \\
&= \sum_{i=1}^n \iota_i\pi_i'\iota_i'\pi_i \text{ since the canonical mappings satisfy} \\
&\qquad\qquad \pi_i\iota_j = 0 \text{ for } i \neq j \text{ as shown above} \\
&= \sum_{i=1}^n \iota_i 1_{A_i}\pi_i \text{ since } \pi_i'\iota_i' = 1_{A_i} \text{ by above} \\
&= \sum_{i=1}^n \iota_i\pi_i = 1_{A_i} \text{ by (iii) above.}
\end{aligned}$$

# Theorem IV.1.14 (continued 3)

**Proof (continued).** Similarly,

$$\begin{aligned}\psi\varphi &= \sum_{i=1}^{n}\sum_{j=1}^{n} \iota'_i \pi_i \iota_j \pi'_j \\ &= \sum_{i=1}^{n} \iota'_i \pi'_i \text{ since } \pi_i \text{ and } \iota_i \text{ satisfy (i), (ii), (iii)} \\ &= 1_{A_1 \oplus A_2 \oplus \cdots \oplus A_n}.\end{aligned}$$

By Theorem I.2.3(ii), $\varphi$ is a group isomorphism and so $A \cong A_1 \oplus A_2 \oplus \cdots \oplus A_n$. □

# Theorem IV.1.14 (continued 3)

**Proof (continued).** Similarly,

$$\begin{aligned}\psi\varphi &= \sum_{i=1}^{n}\sum_{j=1}^{n} \iota_i'\pi_i\iota_j\pi_j' \\ &= \sum_{i=1}^{n} \iota_i'\pi_i' \text{ since } \pi_i \text{ and } \iota_i \text{ satisfy (i), (ii), (iii)} \\ &= 1_{A_1\oplus A_2\oplus\cdots\oplus A_n}.\end{aligned}$$

By Theorem I.2.3(ii), $\varphi$ is a group isomorphism and so $A \cong A_1 \oplus A_2 \oplus \cdots \oplus A_n$. □

# Lemma IV.1.17

**Lemma IV.1.17. The Short Five Lemma.**
Let $R$ be a ring and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

a commutative diagram of $R$-modules and $R$-module homomorphisms such that each row is a short exact sequence. Then

(i) if $\alpha$ and $\gamma$ are monomorphisms then $\beta$ is a monomorphism;
(ii) if $\alpha$ and $\gamma$ are epimorphisms then $\beta$ is an epimorphism;
(iii) if $\alpha$ and $\gamma$ are isomorphisms then $\beta$ is an isomorphism.

**Proof. (i)** Let $b \in B$ and suppose $\beta(b) = 0$. By Theorem I.2.3 (see the comment on page 170) the result follows if we show that $b = 0$.

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a commutative diagram of $R$-modules and $R$-module homomorphisms such that each row is a short exact sequence. Then

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(iii) if $\alpha$ and $\gamma$ are isomorphisms then $\beta$ is an isomorphism.

**Proof. (i)** Let $b \in B$ and suppose $\beta(b) = 0$. By Theorem I.2.3 (see the comment on page 170) the result follows if we show that $b = 0$.

# Lemma IV.1.17 (continued 1)

**Proof (continued). (i)** We have

$$\begin{aligned} \gamma g(b) &= g'\beta(b) \text{ by the commutivity} \\ &= g'(0) \text{ since } \beta \text{ is a homomorphism} \\ &= 0 \text{ since } g' \text{ is a homomorphism.} \end{aligned}$$

This implies $g(b) = 0$ since $\gamma$ is hypothesized to be one to one. So $b \in \text{Ker}(g)$. Since the top row is a (short) exact sequence, then $\text{Im}(f) = \text{Ker}(g)$ and so $b = f(a)$ for some $a \in A$.

# Lemma IV.1.17 (continued 1)

**Proof (continued). (i)** We have

$$\begin{aligned} \gamma g(b) &= g'\beta(b) \text{ by the commutivity} \\ &= g'(0) \text{ since } \beta \text{ is a homomorphism} \\ &= 0 \text{ since } g' \text{ is a homomorphism.} \end{aligned}$$

This implies $g(b) = 0$ since $\gamma$ is hypothesized to be one to one. So $b \in \text{Ker}(g)$. Since the top row is a (short) exact sequence, then $\text{Im}(f) = \text{Ker}(g)$ and so $b = f(a)$ for some $a \in A$. We have

$$\begin{aligned} f'\alpha(a) &= \beta f(a) \text{ by commutivity} \\ &= \beta(b) \text{ since } f(a) = b \\ &= 0 \text{ by hypothesis.} \end{aligned}$$

# Lemma IV.1.17 (continued 1)

**Proof (continued). (i)** We have

$$\begin{aligned}\gamma g(b) &= g'\beta(b) \text{ by the commutivity} \\ &= g'(0) \text{ since } \beta \text{ is a homomorphism} \\ &= 0 \text{ since } g' \text{ is a homomorphism.}\end{aligned}$$

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Since the bottom row is a short exact sequence then, by the note above, $f'$ is one to one and so the only thing mapped to 0 by $f'$ is 0 and we must have $\alpha(a) = 0$.

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**Proof (continued). (i)** We have

$$\begin{aligned} \gamma g(b) &= g'\beta(b) \text{ by the commutivity} \\ &= g'(0) \text{ since } \beta \text{ is a homomorphism} \\ &= 0 \text{ since } g' \text{ is a homomorphism.} \end{aligned}$$

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Since the bottom row is a short exact sequence then, by the note above, $f'$ is one to one and so the only thing mapped to 0 by $f'$ is 0 and we must have $\alpha(a) = 0$.

# Lemma IV.1.17 (continued 2)

**Proof (continued). (i)** But $\alpha$ is one to one by hypothesis and so $a = 0$. Hence $b = f(a) = f(0) = 0$ since $f$ is a homomorphism. So $b = 0$ and $\beta$ is one to one and (i) follows.

**(ii)** Let $b' \in B'$. Then $g'(b') \in C'$. Since $\gamma$ is hypothesized to be onto then $g'(b') = \gamma(c)$ for some $c \in C$. Since the top row is a short exact sequence then, by the not above, $g$ is an epimorphism (onto). Hence $c = g(b)$ for some $b \in B$.

# Lemma IV.1.17 (continued 2)

**Proof (continued). (i)** But $\alpha$ is one to one by hypothesis and so $a = 0$. Hence $b = f(a) = f(0) = 0$ since $f$ is a homomorphism. So $b = 0$ and $\beta$ is one to one and (i) follows.

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$$\begin{aligned} g'\beta(b) &= \gamma g(b) \text{ by commutivity} \\ &= \gamma(c) \text{ since } c = g(b) \\ &= g'(b') \text{ since } g'(b') = \gamma(c). \end{aligned}$$

# Lemma IV.1.17 (continued 2)

**Proof (continued). (i)** But $\alpha$ is one to one by hypothesis and so $a = 0$. Hence $b = f(a) = f(0) = 0$ since $f$ is a homomorphism. So $b = 0$ and $\beta$ is one to one and (i) follows.

**(ii)** Let $b' \in B'$. Then $g'(b') \in C'$. Since $\gamma$ is hypothesized to be onto then $g'(b') = \gamma(c)$ for some $c \in C$. Since the top row is a short exact sequence then, by the not above, $g$ is an epimorphism (onto). Hence $c = g(b)$ for some $b \in B$. We have

$$\begin{aligned} g'\beta(b) &= \gamma g(b) \text{ by commutivity} \\ &= \gamma(c) \text{ since } c = g(b) \\ &= g'(b') \text{ since } g'(b') = \gamma(c). \end{aligned}$$

Thus $0 = g'\beta(b) - g'(b') = g'(\beta(b) - b')$ and $\beta(b) - b' \in \text{Ker}(g') = \text{Im}(f')$ by the exactness of the bottom row. Say $f'(a')\beta(b) - b'$ where $a' \in A$. Since $\alpha$ is hypothesized to be onto, then $\alpha(a) = a'$ for some $a \in A$.

# Lemma IV.1.17 (continued 2)

**Proof (continued). (i)** But $\alpha$ is one to one by hypothesis and so $a = 0$. Hence $b = f(a) = f(0) = 0$ since $f$ is a homomorphism. So $b = 0$ and $\beta$ is one to one and (i) follows.

**(ii)** Let $b' \in B'$. Then $g'(b') \in C'$. Since $\gamma$ is hypothesized to be onto then $g'(b') = \gamma(c)$ for some $c \in C$. Since the top row is a short exact sequence then, by the not above, $g$ is an epimorphism (onto). Hence $c = g(b)$ for some $b \in B$. We have

$$\begin{aligned} g'\beta(b) &= \gamma g(b) \text{ by commutivity} \\ &= \gamma(c) \text{ since } c = g(b) \\ &= g'(b') \text{ since } g'(b') = \gamma(c). \end{aligned}$$

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# Lemma IV.1.17 (continued 3)

**Proof (continued). (ii)** Consider $b - f(a) \in B$:
$\beta(b - f(a)) = \beta(b) - \beta f(a)$. We have

$$\begin{aligned} \beta f(a) &= f'\alpha(a) \text{ by commutivity} \\ &= f'(a') \text{ since } a' = \alpha(a) \\ &= \beta(b) - b' \text{ since } f'(a') = \beta(b) - b'. \end{aligned}$$

# Lemma IV.1.17 (continued 3)

**Proof (continued). (ii)** Consider $b - f(a) \in B$:
$\beta(b - f(a)) = \beta(b) - \beta f(a)$. We have

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Hence

$$\begin{aligned} \beta(b - f(a)) &= \beta(b) - \beta f(a) \\ &= \beta(b) - (\beta(b) - b') \text{ by the previous computation} \\ &= b'. \end{aligned}$$

Since $b' \in B'$ was arbitrary, then $\beta$ is onto (an epimorphism) and (ii) follows.

# Lemma IV.1.17 (continued 3)

**Proof (continued). (ii)** Consider $b - f(a) \in B$:
$\beta(b - f(a)) = \beta(b) - \beta f(a)$. We have

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**(iii)** This follows from (i) and (ii). □

# Lemma IV.1.17 (continued 3)

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Hence

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Since $b' \in B'$ was arbitrary, then $\beta$ is onto (an epimorphism) and (ii) follows.

**(iii)** This follows from (i) and (ii). □

# Theorem IV.1.18

**Theorem IV.1.18.** Let $R$ be a ring and $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ a short exact sequence of $R$-module homomorphisms. Then the following conditions are equivalent:

(i) There is an $R$-module homomorphism $h : A_2 \to B$ with $gh = 1_{A_2}$;

(ii) There is an $R$-module homomorphism $k : B \to A_1$ with $kf = 1_{A_1}$;

(iii) the given sequence is isomorphic (with identity maps on $A_1$ and $A_2$) to the direct sum short exact sequence $\{0\} \to A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to \{0\}$; in particular $B \cong A_1 \oplus A_2$.

**Proof. (i)⇒(iii)** Suppose there is an $R$-module homomorphism $h : A_2 \to B$ with $gh = 1_{A_2}$. Then by Theorem IV.1.13 (with $\psi_1 = f$ and $\psi_2 = h$, where $D = B$) there is a unique module homomorphism $\varphi : A_1 \oplus A_2 \to B$ given by (see the proof of Theorem IV.1.13 where $\varphi(\{a_i\}) = \sum_i \psi_i(a_i)$) the mapping $(a_1, a_2) \mapsto f(a_1) + h(a_2)$.

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# Theorem IV.1.18 (continued 1)

**Proof (continued). (i)⇒(iii)** Consider the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 & \longrightarrow & 0 \\
 & & \Big\downarrow 1_{A_1} & & \Big\downarrow \varphi & & \Big\downarrow 1_{A_2} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0
\end{array}$$

# Theorem IV.1.18 (continued 1)

**Proof (continued). (i)⇒(iii)** Consider the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 & \longrightarrow & 0 \\
 & & \Big\downarrow 1_{A_1} & & \Big\downarrow \varphi & & \Big\downarrow 1_{A_2} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0
\end{array}$$

# Theorem IV.1.18 (continued 2)

**Proof (continued). (i)⇒(iii)** For $a_1 \in A_1$ we have $\varphi\iota_1(a_1) = \varphi(a_1, 0) = f(a_1) + h(0) = f(a_1) = f1_{A_1}(a_1)$ and so $\varphi\iota_1 = fa_{A_1}$.
For $(a_1, a_2) \in A_1 \oplus A_2$ we have

$$\begin{aligned}
1_{A_2}\pi_2(a_1, a_2) &= 1_{A_2}(a_2) = a_2 = 1_{A_2}(a_2) \\
&= gh(a_2) \text{ since } gh = 1_A \text{ by hypothesis} \\
&= gf(a_1) + gh(a_2) \text{ since } df = 0 \text{ by note above} \\
&\qquad \text{(see Remark on p. 176)} \\
&= g(f(a_1) + h(a_1)) \text{ since } g \text{ is a homomorphism} \\
&= g\varphi((a_1, a_2)).
\end{aligned}$$

# Theorem IV.1.18 (continued 2)

**Proof (continued). (i)⇒(iii)** For $a_1 \in A_1$ we have $\varphi\iota_1(a_1) = \varphi(a_1, 0) = f(a_1) + h(0) = f(a_1) = f1_{A_1}(a_1)$ and so $\varphi\iota_1 = fa_{A_1}$. For $(a_1, a_2) \in A_1 \oplus A_2$ we have

$$\begin{aligned} 1_{A_2}\pi_2(a_1, a_2) &= 1_{A_2}(a_2) = a_2 = 1_{A_2}(a_2) \\ &= gh(a_2) \text{ since } gh = 1_A \text{ by hypothesis} \\ &= gf(a_1) + gh(a_2) \text{ since } df = 0 \text{ by note above} \\ &\qquad \text{(see Remark on p. 176)} \\ &= g(f(a_1) + h(a_1)) \text{ since } g \text{ is a homomorphism} \\ &= g\varphi((a_1, a_2)). \end{aligned}$$

So $1_{A_2}\pi = g\varphi$ and the diagram commutes. Since $1_{A_1}$ and $1_{A_2}$ are isomorphisms, then by the Short Five Lemma (Lemma IV.1.17) $\varphi$ is an isomorphism and (iii) holds.

# Theorem IV.1.18 (continued 2)

**Proof (continued). (i)⇒(iii)** For $a_1 \in A_1$ we have $\varphi\iota_1(a_1) = \varphi(a_1, 0) = f(a_1) + h(0) = f(a_1) = f 1_{A_1}(a_1)$ and so $\varphi\iota_1 = fa_{A_1}$. For $(a_1, a_2) \in A_1 \oplus A_2$ we have

$$\begin{aligned}
1_{A_2}\pi_2(a_1, a_2) &= 1_{A_2}(a_2) = a_2 = 1_{A_2}(a_2) \\
&= gh(a_2) \text{ since } gh = 1_A \text{ by hypothesis} \\
&= gf(a_1) + gh(a_2) \text{ since } df = 0 \text{ by note above} \\
&\qquad\qquad \text{(see Remark on p. 176)} \\
&= g(f(a_1) + h(a_1)) \text{ since } g \text{ is a homomorphism} \\
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\end{aligned}$$

So $1_{A_2}\pi = g\varphi$ and the diagram commutes. Since $1_{A_1}$ and $1_{A_2}$ are isomorphisms, then by the Short Five Lemma (Lemma IV.1.17) $\varphi$ is an isomorphism and (iii) holds.

# Theorem IV.1.18 (continued 3)

**Proof. (ii)$\Rightarrow$(iii)** Suppose there is an $R$-module homomorphism $k : B \to A_1$ with $kf = 1_{A_1}$. Then by Theorem IV.1.12 (with $\varphi_1 = k$ and $\varphi_2 = g$, where $C = B$) there is a unique $\psi : B \to A_1 \times A_2 = A_1 \oplus A_2$ (the second equality holding since the indexing set is finite; see page 173) given by (see the proof of Theorem IV.1.12 where $\varphi(c) = \{\varphi_i(c)\}_{i \in I}$) the mapping $\varphi(b) = (k(b), g(b))$.

# Theorem IV.1.18 (continued 3)

**Proof. (ii)⇒(iii)** Suppose there is an $R$-module homomorphism $k : B \to A_1$ with $kf = 1_{A_1}$. Then by Theorem IV.1.12 (with $\varphi_1 = k$ and $\varphi_2 = g$, where $C = B$) there is a unique $\psi : B \to A_1 \times A_2 = A_1 \oplus A_2$ (the second equality holding since the indexing set is finite; see page 173) given by (see the proof of Theorem IV.1.12 where $\varphi(c) = \{\varphi_i(c)\}_{i \in I}$) the mapping $\varphi(b) = (k(b), g(b))$. Consider the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0 \\
 & & \downarrow 1_{A_1} & & \downarrow \varphi & & \downarrow 1_{A_2} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 & \longrightarrow & 0
\end{array}$$

# Theorem IV.1.18 (continued 3)

**Proof. (ii)⇒(iii)** Suppose there is an $R$-module homomorphism $k : B \to A_1$ with $kf = 1_{A_1}$. Then by Theorem IV.1.12 (with $\varphi_1 = k$ and $\varphi_2 = g$, where $C = B$) there is a unique $\psi : B \to A_1 \times A_2 = A_1 \oplus A_2$ (the second equality holding since the indexing set is finite; see page 173) given by (see the proof of Theorem IV.1.12 where $\varphi(c) = \{\varphi_i(c)\}_{i\in I}$) the mapping $\varphi(b) = (k(b), g(b))$. Consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_{A_1}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1_{A_2}} & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 & \longrightarrow & 0 \end{array}$$

# Theorem IV.1.18 (continued 4)

**Proof. (ii)⇒(iii)** For $a_1 \in A_1$ we have $\varphi f(a_1) = (kf(a_1), gf(a_1)) = (a_1, 0)$ (since $kf = 1_{A_1}$ and since $gf = 0$ by the not above [see Remark on page 176]) and $(a_1, 0) = \iota_1(a_1) = \iota_1 1_{A_1}(a_1)$, and so $\varphi f = \iota_1 1_A$. For $b \in B$ we have $1_{A_2} g(b) = g(b) = \pi_2(k(b), g(b)) = \pi_2 \varphi(b)$, and so $1_{A_2} g = \pi_2 \varphi$ and the diagram commutes. Since $1_{A_1}$ and $1_{A_2}$ are isomorphisms, then by the Short Five Lemma (Lemma IV.1.17) $\psi$ is an isomorphism and (iii) holds.

# Theorem IV.1.18 (continued 4)

**Proof. (ii)⇒(iii)** For $a_1 \in A_1$ we have $\varphi f(a_1) = (kf(a_1), gf(a_1)) = (a_1, 0)$ (since $kf = 1_{A_1}$ and since $gf = 0$ by the not above [see Remark on page 176]) and $(a_1, 0) = \iota_1(a_1) = \iota_1 1_{A_1}(a_1)$, and so $\varphi f = \iota_1 1_A$. For $b \in B$ we have $1_{A_2} g(b) = g(b) = \pi_2(k(b), g(b)) = \pi_2 \varphi(b)$, and so $1_{A_2} g = \pi_2 \varphi$ and the diagram commutes. Since $1_{A_1}$ and $1_{A_2}$ are isomorphisms, then by the Short Five Lemma (Lemma IV.1.17) $\psi$ is an isomorphism and (iii) holds.

# Theorem IV.1.18 (continued 5)

**Proof. (iii)⇒(i) and (ii)** Suppose the given sequence $\{0\} \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to \{0\}$ is isomorphic (with identity maps on $A_1$ and $A_2$) to the short exact sequence $\{0\} \to A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to \{0\}$. Let $\varphi : A_1 \oplus A_2 \to B$ be the "center" isomorphism. Consider the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 & \longrightarrow & 0 \\
 & & \downarrow 1_{A_1} & & \downarrow \varphi & & \downarrow 1_{A_2} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0
\end{array}$$

# Theorem IV.1.18 (continued 5)

**Proof. (iii)⇒(i) and (ii)** Suppose the given sequence $\{0\} \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to \{0\}$ is isomorphic (with identity maps on $A_1$ and $A_2$) to the short exact sequence $\{0\} \to A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to \{0\}$. Let $\varphi : A_1 \oplus A_2 \to B$ be the "center" isomorphism. Consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 & \longrightarrow & 0 \\ & & \Big\downarrow 1_{A_1} & & \Big\downarrow \varphi & & \Big\downarrow 1_{A_2} & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0 \end{array}$$

# Theorem IV.1.18 (continued 6)

**Proof (continued). (iii)⇒(i) and (ii)** By the definition of "isomorphic," the diagram commutes. Define $h : A_2 \to B$ as $h = \varphi\iota_2$ and $k : B \to A_1$ as $k = \pi_1\varphi^{-1}$. Now $\pi_i\iota_i = 1_{A_i}$ and $\varphi^{-1}\varphi = 1_{A_1 \oplus A_2}$. Since the diagram commutes, we have

$$\begin{aligned} kf &= (\pi_1\varphi^{-1})f = (\pi_1\varphi^{-1})(f1_{A_1}) \\ &= (\pi_1\varphi^{-1})(\varphi\iota_1) \text{ since } f1_{A_1} = \varphi\iota_1 \text{ by the commutivity of the diagram} \\ &= \pi_1\iota_1 = 1_{A_1} \end{aligned}$$

# Theorem IV.1.18 (continued 6)

**Proof (continued). (iii)⇒(i) and (ii)** By the definition of "isomorphic," the diagram commutes. Define $h : A_2 \to B$ as $h = \varphi\iota_2$ and $k : B \to A_1$ as $k = \pi_1\varphi^{-1}$. Now $\pi_i\iota_i = 1_{A_i}$ and $\varphi^{-1}\varphi = 1_{A_1 \oplus A_2}$. Since the diagram commutes, we have

$$\begin{aligned} kf &= (\pi_1\varphi^{-1})f = (\pi_1\varphi^{-1})(f1_{A_1}) \\ &= (\pi_1\varphi^{-1})(\varphi\iota_1) \text{ since } f1_{A_1} = \varphi\iota_1 \text{ by the commutivity of the diagram} \\ &= \pi_1\iota_1 = 1_{A_1} \end{aligned}$$

and

$$\begin{aligned} gh &= g(\varphi\iota_2) = (f\varphi)\iota_2 \\ &= (1_{A_2}\pi_2)\iota_2 \text{ since } g\varphi = 1_{A_2}\pi_2 \text{ by the commutivity of the diagram} \\ &= 1_{A_2}(\pi_2\iota_2) = 1_{A_2}1_{A_2} = 1_{A_2}. \end{aligned}$$

So (i) and (ii) follow. □

# Theorem IV.1.18 (continued 6)

**Proof (continued). (iii)⇒(i) and (ii)** By the definition of "isomorphic," the diagram commutes. Define $h : A_2 \to B$ as $h = \varphi\iota_2$ and $k : B \to A_1$ as $k = \pi_1\varphi^{-1}$. Now $\pi_i\iota_i = 1_{A_i}$ and $\varphi^{-1}\varphi = 1_{A_1 \oplus A_2}$. Since the diagram commutes, we have

$$\begin{aligned} kf &= (\pi_1\varphi^{-1})f = (\pi_1\varphi^{-1})(f1_{A_1}) \\ &= (\pi_1\varphi^{-1})(\varphi\iota_1) \text{ since } f1_{A_1} = \varphi\iota_1 \text{ by the commutivity of the diagram} \\ &= \pi_1\iota_1 = 1_{A_1} \end{aligned}$$

and

$$\begin{aligned} gh &= g(\varphi\iota_2) = (f\varphi)\iota_2 \\ &= (1_{A_2}\pi_2)\iota_2 \text{ since } g\varphi = 1_{A_2}\pi_2 \text{ by the commutivity of the diagram} \\ &= 1_{A_2}(\pi_2\iota_2) = 1_{A_2}1_{A_2} = 1_{A_2}. \end{aligned}$$

So (i) and (ii) follow. □