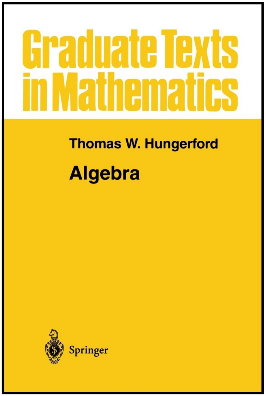
Modern Algebra

Chapter IV. Modules

IV.3. Projective and Injective Modules—Proofs of Theorems



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Theorem IV.3.2

Theorem IV.3.2. Every free module F over a ring with identity is projective.

Proof. By Note IV.3.A, we see that it is sufficient to consider the diagram (right) where A and B are unitary modules, g is an epimorphism, and F a free R-module with, say, basis $X \subset F$. Let $\iota : X \to F$ be the inclusion mapping. For each $x \in X$ we have $f(\iota(x)) \in B$.

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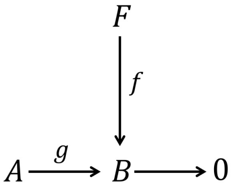

Proof. By Note IV.3.A, we see that it is sufficient to consider the diagram (right) where A and B are unitary modules, g is an epimorphism, and F a free R-module with, say, basis $X \subset F$. Let $\iota : X \to F$ be the inclusion mapping. For each $x \in X$ we have $f(\iota(x)) \in B$. Since g is an epimorphism (and so is surjective), there exists $a_x \in A$ such that $g(ax) = f(\iota(x))$. Since F is a free R-module, then the map $X \to A$ given by $x \mapsto a_x$ induces a unique R-module homomorphism $h : F \to A$ such that $h(c(x)) = a_x$ for all $x \in X$ by Theorem IV.2.1(iv) (with R-module homomorphism $\bar{f}$ there as h here).

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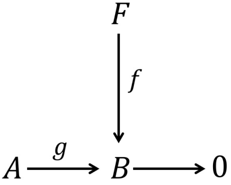
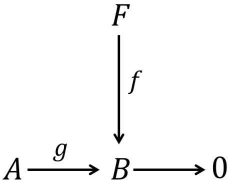


Theorem IV.3.2 (continued)

Theorem IV.3.2. Every free module F over a ring with identity is projective.

Proof (continued). Consequently, $gh\iota(x) = g(a_x) = f(\iota(x))$ for all $x \in X$ so that $gh\iota = f\iota : X \to B$. Since h is unique, then with g given and the inclusion mapping uniquely defined), then $gh\iota$ can be uniquely extended to all of F to give $gh : F \to B$ as the only R-module homomorphism mapping $F \to B$ that takes on the given values on $X \subset F$. Since f also has this property, the $gh = f$. That is, by Definition IV.3.1, F is a projective R-module, as claimed. □



Corollary IV.3.3

Corollary IV.3.3. Every module A over a ring R in the homomorphic image of a projective module.

Proof. By Corollary IV.2.2, A is the homomorphic image of a free R-module F. By Theorem IV.3.2, modified as described in Note IV.3.B, we have that free R-module F is projective, so that A is the homomorphic image of projective module F, as claimed. □

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Theorem IV.3.4

Theorem IV.3.4. Let R be a ring. The following conditions on R-module P are equivalent.

(i) P is projective;

(ii) every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split exact (hence $B \cong A \oplus P$);

(iii) there is a free module F and an R-module K such that $F \cong K \oplus P$.

Proof. (i) $\Rightarrow$ (ii) Consider the diagram (right). Since P is hypothesized to be projective, then g is an epimorphism and there exists R-module homomorphism $h : P \to B$ such that $gh = 1_P$. So the short exact sequence we need for part (ii), $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ satisfies the condition $g^{-1} = h$.

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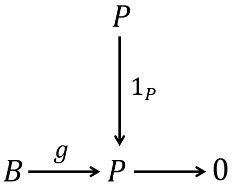

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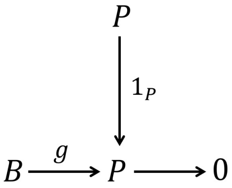

Theorem IV.3.4 (continued 1)

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Proof (continued). Hence $0 \to A \xrightarrow{f} B \underset{h}{\overset{g}{\rightleftarrows}} P \to 0$ where $gh = 1_P$. So by Theorem IV.1.18(i) (Theorem IV.1.18 gives equivalent conditions for a short exact sequence to be split exact), the sequence is split exact, as claimed. By Theorem IV.1.18(iii), $B \cong A \oplus P$, as claimed.

(ii) $\Rightarrow$ (iii) By Corollary IV.2.2, there is a free R-module F (whether we use the term "free" in the sense defined by Theorem IV.2.1 or in the sense of Note IV.2.D and Exercise IV.2.2; see Note IV.2.G) and an epimorphism $g : F \to P$. Define $K = \mathrm{Ker}(g)$.

Theorem IV.3.4 (continued 1)

Theorem IV.3.4. Let R be a ring. The following conditions on R-module P are equivalent.

(ii) every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split exact (hence $B \cong A \oplus P$);

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Proof (continued). Then $0 \to K \xrightarrow{\iota} F \xrightarrow{g} P \to 0$ is an exact sequence (by Definition IV.1.16), and so by Theorem IV.1.18(iii) (where a split exact sequence is defined), we have $F = K \oplus P$, as claimed.

(iii) $\Rightarrow$ **(i)** Since, by hypothesis, $F \cong K \oplus P$ then there is an isomorphism $\varphi : F \to K \oplus P$. Define $\pi : F \to P$ as $\pi = \pi_P \varphi$ with π_P as the canonical projection onto P. Similarly let $\iota : P \to F$ be the composition of the canonical injection mapping $P \to K \oplus P$ with φ^{-1}.

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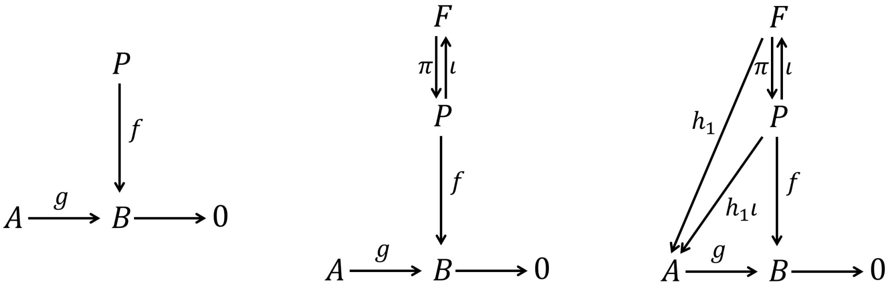
Theorem IV.3.4 (continued 3)

Proof (continued). With the following diagram of R-module homomorphisms (left) where the bottom row is exact, consider the augmented diagram (center).



Since F is a free module by the hypothesis in (iii), then by Theorem IV.3.2 F is projective. We can modify the center diagram by composing f and π so that $f\pi : F \to B$ (and P is not present), as in the right diagram.

Theorem IV.3.4 (continued 4)

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(i) P is projective;

(iii) there is a free module F and an R-module K such that $F \cong K \oplus P$.

Proof (continued). So the projectivity of F then implies that there exists R-module homomorphism $h_1 : F \to A$ such that $gh_1 = f\pi$. Let $h = h_1\iota : P \to A$ (since $\iota : P \to F$ and then $h_1 : F \to A$). Then

$$gh = g(h_1\iota) = (gh_1)\iota = (f\pi)\iota = f(\pi\iota) = f1_P = f.$$

Therefore, by Definition IV.3.1, P is projective, as claimed. □

Proposition IV.3.5

Proposition IV.3.5. Let R be a ring. A direct sum of R-modules $\sum_{i\in I} P_i$ is projective if and only if each P_i is projective.

Proof. Suppose $\sum_{i\in I} P_i$ is projective. Consider the diagram (right) where $\sum_{i\in I} P_i \cong P_j \oplus \sum_{i\neq j} P_i$. This is the same situation we had in the proof of (iii) $\Rightarrow$ (i) of Theorem IV.3.4 with $F = \sum_{i\in I} P_i$, $P = P_j$, and $K = \sum_{i\neq j} P_i$. The proof in Theorem IV.3.4 is only based on the fact that F is projective. Since we have $F = \sum_{i\in I} P_i$ is projective here, then the same proof holds here to prove that P_j is projective. Since j is an arbitrary element of I, we have that P_i is projective for all $i \in I$, as claimed.

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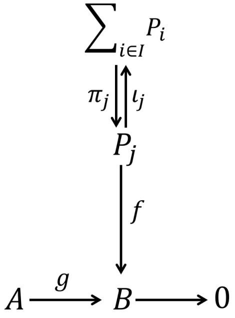

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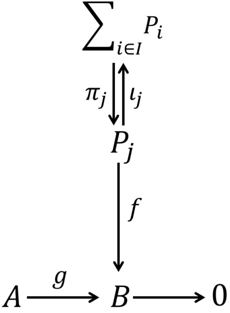

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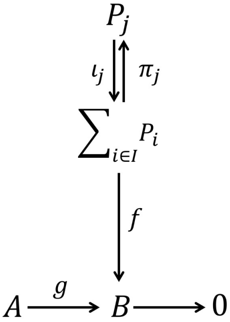
Proof (continued). Suppose P_i is projective for all $i \in I$. Consider the diagram (right). Since P_j is projective, there exists $h_j : P_j \to A$ such that $gh_j = f\iota_j$. By Theorem IV.1.13, there is a unique homomorphism $h : \sum_{i\in I} P_i \to A$ with $h\iota_j = h_j$ for all $j \in I$. Now for p in the direct sum $\sum_{i\in I} P_i$ we have $p(i) = p_i \in P_i$ for finitely many $i \in I$ (WLOG, say $i \in \{1, 2, \ldots, n\}$) and $p(i) = 0 \in P_i$ for the remaining $i \in I$. Write $p = \sum_{k=1}^{n} y_k$ where $y_k(i) = p_i$ if $i = k$ and $y_k(i) = 0$ if $i \neq k$, so that $\iota_k(p_k) = y_k$ for $k \in \{1, 2, \ldots, n\}$.



Proposition IV.3.5 (continued 1)

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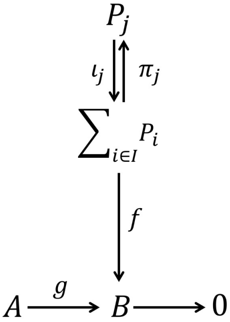
Proof (continued). Suppose P_i is projective for all $i \in I$. Consider the diagram (right). Since P_j is projective, there exists $h_j : P_j \to A$ such that $gh_j = f\iota_j$. By Theorem IV.1.13, there is a unique homomorphism $h : \sum_{i \in I} P_i \to A$ with $h\iota_j = h_j$ for all $j \in I$. Now for p in the direct sum $\sum_{i \in I} P_i$ we have $p(i) = p_i \in P_i$ for finitely many $i \in I$ (WLOG, say $i \in \{1, 2, \ldots, n\}$) and $p(i) = 0 \in P_i$ for the remaining $i \in I$. Write $p = \sum_{k=1}^{n} y_k$ where $y_k(i) = p_i$ if $i = k$ and $y_k(i) = 0$ if $i \neq k$, so that $\iota_k(p_k) = y_k$ for $k \in \{1, 2, \ldots, n\}$.



Proposition IV.3.5 (continued 2)

Proof (continued). Then

$$\begin{aligned} gh(p) &= gh\left(\sum_{k=1}^{n} y_k\right) \text{ since } p = \sum_{k=1}^{n} y_k \\ &= \sum_{k=1}^{n} gh(y_k) \text{ since } g, h \text{ are } R\text{-module homomorphisms} \\ &= \sum_{k=1}^{n} gh(\iota_k p_k) = \sum_{k=1}^{n} gh_k(p_k) \text{ since } h_k = j\iota_k \\ &= \sum_{k=1}^{n} f\iota_k(p_k) \text{ since } gh_k = f\iota_k \\ &= f\left(\sum_{k=1}^{n} \iota_k(p_k)\right) \text{ since } f \text{ is a } R\text{-module homomorphism} \end{aligned}$$

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Proof (continued). ...

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Since p is an arbitrary element of $\sum_{i \in I} P_i$, we have $gh = f$, as needed. Therefore $\sum_{i \in I} P_i$ is projective, as claimed. □