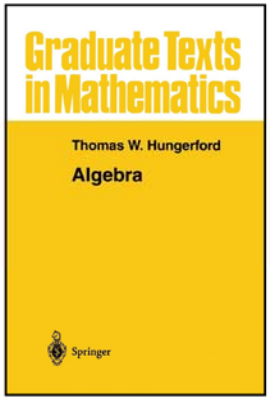
Modern Algebra

Chapter IV. Modules

IV.7. Algebras—Proofs of Theorems (partial)



Theorem IV.7.A

Theorem IV.7.A. If A is a K-algebra and ring A has an identity, then a (left, right, two-sided) ideal of ring A is also a (left, right, two-sided, respectively) algebra ideal of K-algebra A.

Proof. Let J be a left (respectively, right) ideal of ring A. Then J is a subring of A and so J inherits properties (i) and (ii) of the definition of K-algebra (Definition IV.7.1). We just need to verify the left (right) closure $kj \in J$ ($jk \in J$) for all $k \in K$ and $j \in J$. Since A has identity, say 1_A, then for all $k \in K$ and $a \in A$ we have $ka = k(1_A a) = (k1_A)a$ by Definition IV.7.1(ii), and similarly $ak = (a1_A)k = a(1_A k)$. Notice $k1_A \in A$ (respectively, $1_A k \in A$) since $(A, +)$ is a left (respectively, right) K-module. Consequently for $k \in K$ and J a left ideal,

$$\begin{aligned} kJ &= k(1_A J) = (k1_A)J \text{ by Definition IV.7.1(ii)} \\ &\subset J \text{ since } k1_A \in A \text{ and } J \text{ is a left ideal of } A. \end{aligned}$$

Theorem IV.7.A (continued)

Theorem IV.7.A. If A is a K-algebra and ring A has an identity, then a (left, right, two-sided) ideal of ring A is also a (left, right, two-sided, respectively) algebra ideal of K-algebra A.

Proof (continued). Similarly when J is a right ideal,

$$\begin{aligned} Jk &= (J1_A)k = J(1_A k) \text{ by Definition IV.7.1(ii)} \\ &\subset J \text{ since } 1_A k \in A \text{ and } J \text{ is a right ideal of } A. \end{aligned}$$

Therefore, J is a left (respectively, right) algebra ideal of K-module A (and, of course, if J is a two-sided ideal of ring A then J is a two-sided algebra ideal of K-module A). □