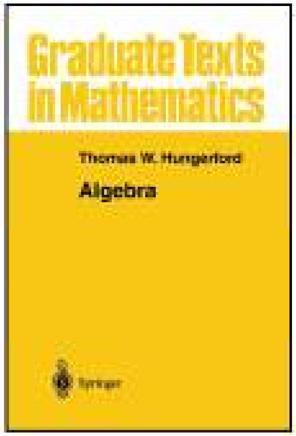
Modern Algebra

Chapter IX. The Structure of Rings

IX.2. The Jacobson Radical—Proofs of Theorems



Lemma IX.2.4

Lemma IX.2.4. If I (where $I \neq R$) is a regular left ideal of a ring R, then I is contained in a maximal left ideal which is regular.

Proof. Since I is a regular left ideal of R, by Definition IX.1.2 of "regular," there is $e \in R$ such that $r - re \in I$ for all $r \in R$. If J is any left ideal of R containing I then for all $r \in R$, $r - re \in I \subset J$ so that J is also a regular left ideal of R. With $I \subset J$ and $e \in J$ we have that $re \in J$ for all $r \in R$, since J is a left ideal of R, and so $r - re \in I \subset J$ implies $r = (r - re) + re \in J$ for every $r \in R$, whence we must have $R = J$. Therefore, if J is a left ideal of R containing I that is not equal to R then $r \notin J$. Let $\mathcal{S}$ be the set of all left ideals L of R such that $I \subset L \subsetneq R$. Put a partial ordering on $\mathcal{S}$ using subset inclusion.

Lemma IX.2.4 (continued)

Lemma IX.2.4. If I (where $I \neq R$) is a regular left ideal of a ring R, then I is contained in a maximal left ideal which is regular.

Proof (continued). For any chain in $\mathcal{S}$, say $\{L_i\}_{i \in K}$ (where K denotes some indexing set), define $L' = \cup_{i \in K} L_i$. As shown in the proof of Theorem III.2.18, L' is a left ideal of R. Since $e \notin L_i$ for all $i \in K$ then $e \notin L'$ so that $L' \subsetneq R$ and so $L' \in \mathcal{S}$. So L' is an upper bound of chain $\{L_i\}_{i \in K}$. So by Zorn's Lemma, $\mathcal{S}$ has a maximal element of M. So M is a maximal left ideal of R and since $I \subset M$ then, as shown above, M is regular, as claimed. □

Lemma IX.2.5

Lemma IX.2.5. Let R be a ring and let K be the intersection of all regular maximal left ideals of R. Then K is a left quasi-regular left ideal of R.

Proof. K is a left ideal by Corollary III.2.3. For $a \in K$, define $T = \{r + ra \mid r \in R\}$. If $T = R$ then there exists $r \in R$ such that $r + ra = -a$, or $r + a + ra = 0$ and hence a is left quasi-regular. So K is left quasi-regular if $T = R$ (for arbitrary $a \in K$). T is a left ideal by Theorem III.2.2 (since $(r_1 + r_1 a) - (r_1 + r_2 a) = (r_1 - r_2) + (r_1 - r_2)a \in T$ and $r(r_1 + r_1 a) = (rr_1) + (rr_1)a \in T$). T is regular with $e = -a$ since for all $r \in R$, $r - re = r - r(-a) = r + ra \in T$. ASSUME $T \neq R$. Then T is a proper regular left ideal of R and by Lemma IX.2.4, $T \subset I_0$ where I_0 is a regular maximal left ideal of R. (Notice that if R has no regular maximal left ideals then $T \neq R$ cannot hold so that $T = R$ in this case, in keeping with the set theoretic convention mentioned in Note IX.2.A.)

Lemma IX.2.5 (continued)

Lemma IX.2.5. Let R be a ring and let K be the intersection of all regular maximal left ideals of R. Then K is a left quasi-regular left ideal of R.

Proof (continued). Since $a \in K \subset I_0$, then $ra \in I_0$ for all $r \in R$ (since I_0 is a left ideal of R). Thus since $r + ra \in T \subset I_0$, we must have $(r + ra) - ra = r \in I_0$ for all $r \in R$. Consequently, $R = I_0$. But this CONTRADICTS the fact that I_0 is a maximal left ideal of R (so that $I_0 \neq R$ by Definition II.2.17). So the assumption that $T \neq R$ is false and hence $T = R$. Hence K is left quasi-regular, as explained above. □

Lemma IX.2.6

Lemma IX.2.6. Let R be a ring that has a simple left R-module. If I is a left quasi-regular left ideal R, then I is contained in the intersection of all the left annihilators of simple left R-modules.

Proof. ASSUME $I \not\subset \cap\mathcal{A}(A)$, where the intersection is taken over all simple left R-modules A. Then I is not the annihilator of some simple left R-module B and so $IB \neq \{0\}$, whence $Ib \neq \{0\}$ for some nonzero $b \in B$. Since I is a left ideal then Ib is a nonzero submodule of B (see Definition IV.1.1 or "module"). Since B is simple, $B = Ib$ and hence $ab = -b$ for some $a \in I$. Since I is left quasi-regular, by Definition IX.2.2, there exists $r \in R$ such that $r + a + ra = 0$. Therefore

$$0 = 0b = (r + a + ra)b = rb + ab + rab = rb - b - rb = -b$$

and so $b = 0$, a CONTRADICTION to the fact that b is nonzero. So the assumption that $I \not\subset \cap\mathcal{A}(A)$ is false and it must be that $I \subset \cap\mathcal{A}(A)$, as claimed. □

Lemma IX.2.7

Lemma IX.2.7. An ideal P of a ring R is left primitive if and only if P is the left annihilator of a simple left R-module.

Proof. Suppose P is a left primitive ideal. Then by definition (Definition IX.2.1), ring R/P is a left primitive ring. So by Definition IX.1.5 ("primitive ring"), there is a simple faithful left R/P-module A (so $\mathcal{A}(A) = \{0\}$; that is, $(r + P)A = \{0\}$ if and only if $r + P$ is the additive identity in R/P). We claim that A is an R-module with ra defined as $(r + P)a$ for $r \in R$, $a \in A$, and $r + P \in R/P$. For $r, s \in R$ and $a, b \in A$ we have

(i) $r(a + b) = (r + P)(a + b) = (r + P)a + (r + P)b = ra + rb$,

(ii) $(r + s)a = ((r + s) + P)a = ((r + P) + (s + P)a = (r + P)a + (s + P)a = ra + sa$, and

(iii) $r(sa) = r((s + P)a) = (r + P)((s + P)a) = (r + P)(s + P))a = (rs + P)a = (rs)a$,

so by Definition IV.1.1 of "R-module," A is an R-module.

Lemma IX.2.7 (continued 1)

Proof (continued). Notice that by our definition of ra, we have $RA = (R/P)A$. Since A is a simple R/P-module then $(R/P)A \neq \{0\}$ and so $RA \neq \{0\}$. So every R-module of A is an R/P-submodule of A (with our definition of ra. But A is a simple R/P-module, whence A is a simple R-module. If $r \in R$, then $rA = \{0\}$ if and only if $(r + P)A = \{0\}$. But $(r + P)A = \{0\}$ if and only if $r + P$ is the additive identity in R/p (this is where the fact that A is faithful and $\mathcal{A}(A) = \{0\}$ is used); that is, if and only if $r \in P$. So the left annihilator of R-module A is P. That is, P is the left annihilator of *some* simple left R-module (namely A, a simple faithful left R/P-module).

Conversely, suppose P is the left annihilator of a simple R-module B. We claim that B is a simple R/P-module with $(r + P)b$ defined as rb for $r \in R$, $b \in B$, and $r + P \in R/P$.

Lemma IX.2.7 (continued 2)

Proof (continued). For $r+P, s+P \in R/P$ and $a, b \in B$ we have

(i) $(r+P)(a+b) = r(a+b) = ra + rb = (r+P)a + (r+P)b$,

(ii) $((r+P)+(s+P))b = ((r+s)+P)b = (r+s)b = rb + sb = (r+P)b + (s+P)b$, and

(iii) $(r+P)((s+P)b) = (r+P)(sb) = r(sb) = (rs)b = (rs+P)b = ((r+P)(s+P))b$,

so by Definition IV.1.1, A is an R/P-module. As above, $RB = (R/P)B \neq \{0\}$ (since B is a simple R-module by hypothesis), so every R/P-submodule of B is an R-submodule of B. Since B is a simple R-module then B is a simple R/P-module. Furthermore, since P is the left annihilator of B then $(r+P)B = \{0\}$ implies $rB = \{0\}$ (with our definition of $(r+P)b$ in R/P-module B) and so $r \in \mathcal{A}(A) = P$. Then $r+P = P$ (the additive identity in R/P; that is, $r+P = 0$ in R/P).

Lemma IX.2.7 (continued 3)

Lemma IX.2.7. An ideal P of a ring R is left primitive if and only if P is the left annihilator of a simple left R-module.

Proof (continued). So in R/P-module B, B is simple and the left annihilator of B is $\mathcal{A}(B) = \{0\}$. That is, B is a faithful R/P-module. Therefore R/P is a left primitive ring by Definition IX.1.5, whence (by Definition IX.2.1) P is a left primitive ideal of R, as claimed. □

Lemma IX.2.8

Lemma IX.2.8. Let I be a left ideal of ring R. If I is left quasi-regular, then I is right quasi-regular.

Proof. Since I is left quasi-regular then by Definition IX.2.2, for $a \in I$ there exists $r \in R$ such that $r \circ a = r + a + ra = 0$. Since I is a left ideal, $ra \in I$ and hence $r = -a - ra \in I$. Again since I is left quasi-regular then there is $s \in R$ such that $s \circ r = s + r + sr = 0$, whence s is *right* quasi-regular. Consequently,

$$a = 0 + a + 0a = 0 \circ a = (s \circ r) \circ a = (s \circ r) + a + (s \circ r)a$$

$$= s+r+sr+a+(s+r+sr)a = s+r+sr+a+sa+ra+sra = s+(r+a+rs)$$

$$+s(r+a+ra) = s \circ (r+a+ra) = s \circ (r \circ a) = s \circ 0 = 0 + s + 0s = s.$$

So a is a right quasi-regular element of I. Since a is an arbitrary element of I then I is a right quasi-regular left ideal of R. □

Theorem IX.2.3

Theorem IX.2.3. If R is a ring, then there is an ideal $J(R)$ of R such that:

(i) $J(R)$ is the intersection of all the left annihilators of simple left R-modules;

(ii) $J(R)$ is the intersection of all the regular maximal left ideals of R;

(iii) $J(R)$ is the intersection of all the left primitive ideals of R;

(iv) $J(R)$ is a left quasi-regular left ideal which contains every left quasi-regular left ideal of R;

(v) Statements (i)–(iv) are also true if "left" is replaced by "right."

Proof. Let $J(R)$ be the intersection of all left annihilators of simple left R-modules (so

$$\text{if } R \text{ has no simple left } R\text{-modules then } J(R) = R \quad (*)$$

by the set theoretic convention of Note IX.2.A).

Theorem IX.2.3 (continued 1)

Proof (continued). Notice that we are sort of taking part (i) as the definition of the Jacobson radical and proving that (ii) and (iii) are equivalent classifications of $J(R)$. $J(R)$ is an ideal of R by Theorem IX.1.4 (which says that annihilator of modules are ideals) and Corollary III.2.3 (which says that intersections of ideals are ideals).

We first observe that R itself cannot be the annihilator of a simple left R-module A, for this would imply that $RA = \{0\}$, in contradiction to the definition of "simple left R-module." So if $J(R) = R$ then R has no simple left R-modules. This, combined with $(*)$, gives that the following are equivalent:

(a) $J(R) = R$.

(b) R has no simple left R-modules.

By Theorem IX.1.3, A is a simple left R-module if and only if $A \cong R/I$ for some regular maximal left ideal I of R. So (b) holds if and only if:

(c) R has no regular maximal left ideals.

Theorem IX.2.3 (continued 2)

Proof (continued). By Lemma IX.2.7, we have a left annihilator of a simple left R-module if and only if we have a left primitive ideal of R. So (b) *no simple left R-modules) is equivalent to:

(d) R has no left primitive ideals.

Therefore, by the set theoretic convention of Note IX.2.A, (ii), (iii), and (iv) hold if $J(R) = R$. So for the remainder of the proof we may assume $J(R) \neq R$.

(ii) Let K be the intersection of all the regular left ideals of R. We want to show $J(R) = K$ so that (i) is equivalent (ii). By Lemma IX.2.5, K is a left quasi-regular left ideal of R. Then by Lemma IX.2.6 (notice that R has a simple left R-module since (a) is equivalent to (b) and we are supposing (a) does not hold), K is contained in the intersection of all left annihilators of simple left R-modules so that $K \subset J(R)$. Now suppose $c \in J(R)$.

Theorem IX.2.3 (continued 3)

Proof (continued). Since $J(R)$ is by our definition the intersection of all left annihilators of simple left R-modules, by Theorem IX.1.3 (which implies "simple left R-modules" if and only if "R/I for some regular maximal left ideal")

$$J(R) \text{ is the intersection of the left annihilators of the quotients } R/I, \quad (**)$$

where I runs over all regular maximal left ideals of R. (Notice that a simple left R-module A is *isomorphic* to R/I, but that the annihilator of both A and R/I are elements of R due to our definition of $(r+I)a = ra$ for $r \in R$, $r + I \in R/I$, and $a \in I$ as introduced in Lemma IX.2.7.) For each regular maximal left ideal I (by Definition IX.1.2) ther exists $e \in R$ such that $c - ce \in I$. Since $c \in J(R) \subset \mathcal{A}(R/I)$ by $(**)$, then $c(r+I) = I$ ($I = 0$ in R/I) for all $r \in R$ and so $cr \in I$ for all $r \in R$. In particular, $ce \in I$, and since $c - ce \in I$ then $c \in I$. Since c is an arbitrary element of $J(R)$ and I is an arbitrary regular maximal left ideal of R, then $J(R) \subset \cup I = K$. Therefore $J(R) = K$ and so (ii) is equivalent to (i).

Theorem IX.2.3 (continued 4)

Proof (continued). (iii) Lemma IX.2.7 gives a one to one correspondence between left primitive ideals of R and left annihilator of simple left R-modules, so (i) is equivalent to (iii) (and hence by above, also equivalent to (ii)).

(iv) By (ii), $J(R)$ is the intersection of all the regular maximal left ideals of R. So by Lemma IX.2.5, $J(R)$ is a left quasi-regular left ideal of R. Now we are assuming $J(R) \neq R$ in this case (we showed above that Theorem IX.2.3 holds when $J(R) = R$), so in this case $J(R)$ has a simple left R-module (see the discussion above). By Lemma IX.2.6, based on our definition of $J(R)$, $J(R)$ contains every left quasi-regular left ideal of R. So (iv) holds.

To complete the proof, we must show that (i)–(iv) are true with "right" in the place of "left." Let $J_1(R)$ be the intersection of all right annihilators of all simple right R-modules.

Theorem IX.2.3 (continued 5)

Proof (continued). Since Lemmas IX.2.4 through IX.2.8 hold with "left" and "right" interchanged (see Hungerford's comment on page 426), then the preceding proof holds for $J_1(R)$ (so we need to establish that $J_1(R) = J(R)$). Since $J(R)$ is a left quasi-regular ideal of R by (iv) above, then by Lemma IX.2.8, $J(R)$ is also right quasi-regular. Hence by our definition of $J_1(R)$, $J(R) \subset J_1(R)$. Similarly, $J_1(R)$ is right quasi-regular by (iv) (modified with "left" and "right" interchanged) $J_1(R)$ is also left quasi-regular. So by our definition of $J(R)$, $J_1(R) \subset J(R)$ and hence $J(R) = J_1(R)$, as needed. □

Theorem IX.2.A

Theorem IX.2.A. Let R be a commutative ring with identity which has a unique maximal ideal M (such a ring is a *local ring*; see Definition III.4.12). Then $J(R) = M$.

Proof. By Theorem III.4.13(ii), all nonunits of R are contained in some ideal of R (and this ideal is not equal to R), and since R has a unique maximal ideal then M contains all nonunits of R. By Note IX.2.B, $J(R) \neq R$. By Theorem III.3.2, u is a unit in R if and only if $u \mid r$ for all $r \in R$ so that the only ideal containing u is R itself. So proper ideals of R can only contain nonunits of R and so $J(R)$ contains only nonunits so that $J(R) \subset M$. On the other hand, if $r \in M$ then $1_R + r \notin M$ (otherwise $q_R \in M$ and then $M = R$, but $M \neq R$). Consequently $q_R + r$ is a unit and by Exercise IX.2.1(c), element r is left quasi-regular and so ideal M is left quasi-regular. By Theorem IX.2.3(iv), $J(R)$ contains every quasi-regular left ideal of R, so $M \subset J(R)$. Therefore $J(R) = M$, as claimed. □

Theorem IX.2.10

Theorem IX.2.10. Let R be a ring.

(i) If R is primitive, then R is semisimple.

(ii) If R is simple and semisimple, then R is primitive.

(iii) If R is simple, then R is either a primitive semisimple ring or a radical ring.

Proof. (i) If R is primitive, then (by Definition IX.1.5) R has a simple faithful left R-module A; that is, A is a simple left R-module and its left annihilator satisfies $\mathcal{A}(A) = \{0\}$. By Theorem IX.2.3(i), $J(R) \subset \mathcal{A}(A) = \{0\}$. So $J(R) = \{0\}$ and R is semisimple.

(ii) Let R be simple and semisimple. Then $R \neq \{0\}$ since R is simple. If there is no simple left R-module then by Theorem IX.2.3(i) (and Note IX.2.A), $J(R) = R \neq \{0\}$; but this contradicts the semisimplicity of R. So there is some simple left R-module A. The left annihilator $\mathcal{A}(A)$ is an ideal of R by Theorem IX.1.4.

Theorem IX.2.10 (continued)

Theorem IX.2.10. Let R be a ring.

(i) If R is primitive, then R is semisimple.

(ii) If R is simple and semisimple, then R is primitive.

(iii) If R is simple, then R is either a primitive semisimple ring or a radical ring.

Proof (continued). Since $RA \neq \{0\}$ because A is a *simple* left R-module (see Definition IX.1.1, "simple left module") then $\mathcal{A}(A) \neq R$. Also by Definition IX.1.1, the simplicity of R then implies that $\mathcal{A}(A) = \{0\}$. So (by Definition IX.1.5) A is faithful (and so a simple faithful R-module) and R is primitive.

(iii) Let R be simple. Then ideal $J(R)$ of R is either R or $\{0\}$. If $J(R) = R$ then R is a radical ring. If $J(R) = \{0\}$ then R is semisimple and so, by part (ii), R is primitive. □

Theorem IX.2.B

Theorem IX.2.B. Let D be a division ring. Then the ring of all $n \times n$ matrices over D, $\mathrm{Mat}_n(D)$, is semisimple.

Proof. By Theorem VII.1.4, $\mathrm{Mat}_n(D)$ is isomorphic to the ring of endomorphisms $\mathrm{Hom}_{D'}(V, V)$ where V is a (left) vector space over some division ring D'. In the Example after Definition IX.1.5, it is shown that $R = \mathrm{Hom}_{D'}(V, V)$ is a primitive ring. So by Theorem IX.1.10(i), $R = \mathrm{Hom}_{D'}(V, V)$ is semisimple. Since $R \cong \mathrm{Mat}_n(D)$, then $\mathrm{Mat}_n(D)$ is semisimple. □

Theorem IX.2.12

Theorem IX.2.12. If R is a ring, then every nil right or left ideal is contained in the Jacobson radical $J(R)$.

Proof. Let a be in a nil ideal. Then $a^n = 0$ for some $n \in \mathbb{N}$. Let $r = -a + a^2 - a^3 + \cdots + (-1)^{n-1}a^{n-1}$. Then

$$r + a + ra = (-a + a^2 - a^3 + \cdots + (-1)^{n-1}a^{n-1}) + a$$

$$+(-a + a^2 - a^3 + \cdots + (-1)^{n-1}a^{n-1})a$$

$$= (a^2 - a^3 + \cdots + (-1)^{n-1}a^{n-1}) + (-a^2 + a^3 - \cdots + (-1)^{n-1}a^n = (-1)^{n-1}a^n = 0$$

and similarly $r + a + ar = 0$. Hence a is both left and right quasi-radical. So the nil ideal is a quasi-regular ideal. By Theorem IX.2.3(iv) and (v), the nil ideal is contained in $J(R)$. □

Theorem IX.2.13

Proposition IX.2.13. If R is a left (right) Artinian ring, then the radical $J(R)$ is a nilpotent ideal. Consequently every nil left or right ideal of R is niplotent and $J(R)$ is the unique maximal nilpotent left (right) ideal of R.

Proof. Let $J = J(R)$ and consider $J, J^2, J^3, \ldots$ (again, J^k is the set of all sums of products of k elements of J). Since $J(R)$ is an ideal of R by Theorem IX.2.3, then for any

$$a = (a_{1,1}a_{2,1} \cdots a_{k,1}) + (a_{1,2}a_{2,2} \cdots a_{k,2}) + \cdots + (a_{1,n}a_{2,n} \cdots a_{k,n}) \in J^k$$

and for any $r \in R$ we have

$$ra = ((ra_{1,1})a_{2,1} \cdots a_{k,1}) + ((ra_{1,2})a_{2,2} \cdots a_{k,2}) + \cdots + ((ra_{1,n})a_{2,n} \cdots a_{k,n}) \in J^k$$

since J is an ideal of R and so $ra_{1,i} \in J$ for $i = 1, 2, \ldots, n$. Since J is an ideal (and hence a subring) of R then $a_1a_2a_3 \cdots a_k = (a_1a_2)a_3 \cdots a_k = a'a_3 \cdots a_k$ for some $a' \in J$ and so $J^{k-1} \supset J^k$ for each $k = 2, 3, \ldots$.

Theorem IX.2.13 (continued 1)

Proof (continued). Similarly, any product of two products of k elements of J can be written as a product of k elements of J:

$$(a_1a_2 \cdots a_k)(b_1b_2 \cdots b_k) = a_1a_2 \cdots a_{k-1}(a_kb_1)b_2 \cdots b_k$$

$$= a_1a_2 \cdots (a_{k-1}b')b_2 \cdots b_k = a_1a_2 \cdots a_{k-1}b''b_2 \cdots b_k$$

$$= \cdots = b^{(k)}b_2 \cdots b_k \in J^k,$$

so J^k is closed under products and is a subring of R. Hence $J \supset J^2 \supset J^3 \supset \cdots$ is a descending chain of (left) ideals of R. By hypothesis there exists $k \in \mathbb{N}$ such that $J^i = J^k$ for all $i \geq k$. We claim $J^k = \{0\}$. ASSUME $J^k \neq \{0\}$. Then the set S of all left ideals I such that $J^kI \neq \{0\}$ contains $I = J^k$ since $J^kJ^k = J^{2k} = J^k \neq \{0\}$. By Theorem VIII.1.4, set S has a minimal element $I_0 \in S$. Since $I_0 \in S$ then $J^kI_0 \neq \{0\}$, so there is nonzero $a \in I_0$ such that $J^ka \neq \{0\}$.

Theorem IX.2.13 (continued 2)

Proof (continued). Since J^k is a subring of R then $J^k a$ is a subring of R (it is "clearly" closed under addition and multiplication; notice $a \in I_0 \subset J^k$) and since J^k is a left ideal of R (so $rJ^k \subset J^k$ for all $r \in R$) then $J^k a$ is a left ideal of R (for $ja \in J^k a$ and $r \in R$, $r(ja) = (rj)a = j'a \in J^k a$ for some $j' \in J^k$). Since $I_0 \in S$ is a left ideal of R and $a \in I_0$ then $J^k a \subset I_0$. Furthermore, since $J^k(J^k a) = J^{2k} a = J^k a \neq \{0\}$ then $J^k a \in S$. Consequently, since I_0 is a minimal element of S, $J^k a \in S$, and $J^k a \subset I_0$, then $J^k a = I_0$. Thus for some nonzero $r \in J^k$, $ra = a$. Since J^k is a ring, $-r \in J^k \subset J = J(R)$ and by Theorem IX.2.3(iv) all elements of $J(R)$ are quasi-regular, then $-r$ is quasi-regular. Whence $s - r - sr = 0$ (by Definition IX.2.2) for some $s \in R$. Consequently (using $ra = a$):

$$a = ra = -(-ra) = -(-ra + 0) = -(-ra + sa - sa)$$
$$= -(-ra + sa - s(ra)) = -(-r + s - sr)a = -0a = 0.$$

But by choice, a is nonzero so this is a CONTRADICTION.

Theorem IX.2.13 (continued 3)

Proposition IX.2.13. If R is a left (right) Artinian ring, then the radical $J(R)$ is a nilpotent ideal. Consequently every nil left or right ideal of R is niplotent and $J(R)$ is the unique maximal nilpotent left (right) ideal of R.

Proof (continued). So the assumption that $J^k \neq \{0\}$ is false and hence $J^k = \{0\}$. So $J(R) = J \supset J^2 \supset \cdots \supset J^k = \{0\}$ and so J is a nilpotent ideal of R (by Definition IX.2.11), as claimed.

By Theorem IX.2.12, every nil left or right ideal of R is contained in $J(R)$. Since we have shown $J(R)$ to be nilpotent, then every nil left or right ideal of R is also nilpotent as claimed. Also, since $J(R)$ contains all nil left or right ideals of R and i fI is any nilpotent ideal then I is also a nil ideal (by definition IX.2.11 and the Note after it) and so $I \subset J(R)$. Hence $J(R)$ is the unique maximal nilpotent left (or right) ideal of R, as claimed. □

Theorem IX.2.14

Theorem IX.2.14. If R is a ring, then the quotient ring $R/J(R)$ is semisimple.

Proof. Let $\pi : R \to R/J(R)$ be the canonical epimorphism mapping $r \mapsto r + J(R)$. Denote $\pi(r) = r + J(R) = \overline{r}$ for each $r \in R$. Let $\mathcal{C}$ be the set of all regular maximal left ideals of R. Then by Theorem IX.2.3(ii), $J(R) = \cap_{I \in \mathcal{C}} I$, so if $I \in \mathcal{C}$ then $J(R) \subset I$. By Theorem IV.1.10 (which describes all submodules of $R/J(R)$), $\pi(I) = I/J(R)$ is a maximal left ideal of $R/J(R)$. Since I is regular, there is $e \in R$ such that $r - re \in I$ for all $r \in R$, and $\pi(r - re) = \overline{r} - \overline{r}\,\overline{e} \in \pi(I)$ for all $r \in R$ (and so for all $\overline{R} \in R/J(R)$). Therefore, $\pi(I)$ is regular (by Definition IX.1.2) for every $I \in \mathcal{C}$).

Let $\overline{r} \in \cap_{I \in \mathcal{C}} \pi(I) = \cap_{I \in \mathcal{C}} I/J(R)$. ASSUME $r \notin J(R)$. Then $\overline{R} = r + J(R) \neq J(R)$. So coset $r + J(R) \in I/J(R)$ for all $I \in \mathcal{C}$. Now the cosets of $J(R)$ in I partition I for each $I \in \mathcal{C}$, so these partitions each include $J(R)$ and $r + J(R)$.

Theorem IX.2.14 (continued)

Proof (continued). So cosets in $(\cap_{I \in \mathcal{C}} I)/J(R)$ include both $J(R)$ and $r + J(R)$. However, $\cap_{I \in \mathcal{C}} I = J(R)$, so $(\cap_{I \in \mathcal{C}} I)/J(R) = J(R)/J(R) = \{0\}$ and so $J(R)/J(R)$ includes only the identity coset $J(R)$, a CONTRADICTION. So the assumption that $r \notin J(R)$ is false and it must be that for all $\overline{r} \in \cap_{I \in \mathcal{C}} I = \cap_{I \in \mathcal{C}} I/J(R)$, we have $r \in J(R)$. That is,

$$\cap_{I \in \mathcal{C}} I \subset \pi(J(R)). \qquad (*)$$

Consequently, by applying Theorem IX.2.3(ii) to ring $R/J(R)$, we have that $J(R/J(R))$ is the intersection of all regular maximal left ideals of $R/J(R)$. Since each $\pi(I)$ is a maximal left ideal of $R/J(R)$ for all $I \in \mathcal{C}$ as shown above, then

$$\begin{aligned} J(R/J(R)) &\subset \cap_{I \in \mathcal{C}} \pi(I) \\ &\subset \pi(J(R)) \text{ by } (*) \\ &= J(R)/J(R) = \{0\}. \end{aligned}$$

So $J(R/J(R)) = \{0\}$ and by Definition IX.2.9, $R/J(R)$ is semisimple, as claimed. □

Lemma IX.2.15

Lemma IX.2.15. Let R be a ring and $a \in R$.

(i) If $-a^2$ is left quasi-regular, then so is a.

(ii) $a \in J(R)$ if and only if Ra is a left quasi-regular left ideal.

Proof. (i) If $-a^2$ is left quasi-regular then, by Definition IX.2.2, there is $r \in R$ such that $r + (0a^2) + r(-a^2) = 0$. let $s = r - a - ra$. Then

$$s + a + sa = (r - a - ra) + a + (r - a - ra)a$$
$$= r - a - ra + a + ra - a^2 - ra^2 = r + (-a^2) + r(-a^2) = 0,$$

and so a is left quasi-regular.

(ii) Suppose $a \in J(R)$. Since $J(R)$ is an ideal of R by Theorem IX.2.3, then $Ra \subset J(R)$. Now $J(R)$ is a left quasi-regular left ideal of R by Theorem IX.2.3(iv), so each element of Ra is left quasi-regular; also, Ra is a left ideal of R and so Ra is a left quasi-regular left ideal, as claimed. Conversely, suppose Ra is a left quasi-regular left ideal of R. Consider $K = \{ra + na \mid r \in R, n \in \mathbb{Z}\}$.

Lemma IX.2.15 (continued)

Lemma IX.2.15. Let R be a ring and $a \in R$.

(i) If $-a^2$ is left quasi-regular, then so is a.

(ii) $a \in J(R)$ if and only if Ra is a left quasi-regular left ideal.

Proof (continued). Then $(r_1a + n_1a) - (r_2a + n_2a) = (r_1 - r_2)a + (n_1 - n_2)a \in K$ and for any $r \in R$ we have $r(r_1a + n_1a) = rr_1a + n_ara = (rr_1 + n_qr)a \in K$. So by Theorem III2.2, K is a left ideal of R. Also, $a \in K$ (take $r = 0 \in R$ and $n = 1 \in \mathbb{Z}$) and $Ra \subset K$ (take $n = 0 \in \mathbb{Z}$). If $s = ra + na \in K$ then $-s^2 \in Ra$ since Ra is a ring. So by hypothesis $-s^2$ is left quasi-regular and so, by part (i), s is left quasi-regular. So every element of K is left quasi-regular and so K is a left quasi-regular left ideal of R. Since $a \in K$ then a is left quasi-regular. Since a is an arbitrary element of $J(R)$ then $J(R)$ is left quasi-regular, as claimed. □

Theorem IX.2.16

Theorem IX.2.16.

(i) If an ideal I of a ring R is itself considered as a ring, then $J(I) = I \cap J(R)$.

(ii) If R is semisimple, then so is every ideal of R.

(iii) $J(R)$ is a radical ring.

Proof. (i) Consider $I \cap J(R)$. By Theorem IX.2.3, $J(R)$ is an ideal of R (and so for each $r \in I$, $rJ(R) \subset J(R)$ and $J(R)r \subset J(R)$), so $r(I \cap J(R)) \subset I \cap J(R)$ and $(I \cap J(R))f \subset I \cap J(R)$. That is, $I \cap J(R)$ is an ideal of I. If $a \in I \cap J(R)$ then a is left quasi-regular in R by Theorem IX.2.3(iv), whence $r + a + ra = 0$ for some $r \in R$. But $r = -a - ra \in I$ (since $a \in I$ and I is an ideal of R). Thus every element of $I \cap J(R)$ is left quasi-regular in I (since $r + a + ra = 0$ where $r \in I$). Therefore by Theorem IX.2.3(iv) (applied to ring I), $I \cap J(R) \subset J(I)$.

Theorem IX.2.16 (continued)

Proof (continued). Suppose $a \in J(I)$. For any $r \in R$, $-(ta)^2 = -(rar)a \in IJ(I)$ (since $J(R)$ is a [two-sided] ideal of R) and $IJ(I) \subset J(I)$ (since $J(I)$ is an ideal of I), so that $-(ra)^2 \in J(I)$. Whence, by Theorem IX.2.3(iv) applied to ring I, $-(ra)^2$ is left quasi-regular in I. Consequently, by Lemma IX.2.14(i), ra is left quasi-regular in I, and hence in R (see Definition IX.2.2). Since $r \in R$ is arbitrary, Ra is a left quasi-regular left ideal of R, whence $a \in J(R)$ by Lemma 2.15(ii). Therefore, $a \in J(I) \cap J(R) \subset I \cap J(R)$. Since a is an arbitrary element of $J(I)$, then $J(I) \subset I \cap J(R)$ and, since $I \cap J(R) \subset I$ as shown above, $J(I) = I \cap J(R)$.

(ii) If R is semisimple then, by Definition IX.2.9, $J(R) = \{0\}$. If I is any ideal of R then by part (i), $J(I) \subset I \cap J(R) = \{0\}$ and so I is semisimple.

(iii) Since $I = J(R)$ is an ideal of R by Theorem IX.2.3, then part (i) implies $JJ(R)) = J(R) \cap J(R) = J(R)$. So by Definition IX.2.9 of radical ring, $I = J(R)$ is a radical ring. □

Theorem IX.2.17

Theorem IX.2.17. If $\{R_i \mid i \in I\}$ is a family of rings, then $J\left(\prod_{i\in I} R_i\right) = \prod_{i\in I} J(R_i)$.

Proof. First, we claim that $\{a_i\} \in \prod R_i$ is a left quasi-regular element in $\prod R_i$ if and only if a_i is left quasi-regular in R_i for each $i \in I$. If each a_i is left quasi-regular in R_i then, by Definition IX.2.2, there is $r_i \in R_i$ such that $r_i + a_i + r_i a_i = 0_i$. Then for $\{r_i\} \in \prod R_i$, we have

$$\{r_i\} + \{a_i\} + \{r_i\}\{a_i\} = \{r_i + a_i + r_i a_i\} = \{a_i\} = 0 \in \prod R_i$$

and so $\{a_i\}$ is left quasi-regular in $\prod R_i$. Conversely, if $a = \{a_i\}$ is left quasi-regular in $\prod R_i$, then there is $r = \{r_i\} \in \prod R_i$ such that

$$r + a + ra = \{r_i\} + \{a_i\} + \{r_i\}\{a_i\} = \{r_i + a_i + r_i a_i\} = \{0_i\} = 0 \in \prod R_i.$$

So $r_i + a_i + r_a a_i = 0_i$ for all $i \in I$. That is, a_i is left quasi-regular in R_i for each $i \in I$.

Theorem IX.2.17 (continued)

Proof (continued). Now $J(R_i)$ is a left quasi-regular ideal of R_i by Theorem IX.2.3(iv) (so every element of $J(R_i)$ is left quasi-regular in R_i), so $\prod J(R_i)$ is a left quasi-regular ideal in $\prod R_i$. So by Theorem IX.2.3(iv) again, $\prod J(R_i) \subset J(\prod R_i)$.

For each $k \in I$, let $\pi_k : \prod R_i \to R_k$ be the canonical projection. Consider $I_k = \pi_k(J(\prod R_i))$. Now $J(\prod R_i)$ is an ideal of $\prod R_i$ by Theorem IX.2.3, so $I_k = \pi_k(J(\prod R_i))$ is an ideal of R_k (by Theorem III.2.2, say, where we can use certain closure of $J(\prod R_i)$ in $\prod R_i$ to set the corresponding closure of I_k in R_k). Let $a_k \in I_k$. Then $\{a_i\} \in J(\prod R_i)$, for some $a_i \in R_i$ for $i \in I, i \neq k$. Applying Theorem IX.2.3(iv) to ring $\prod R_i$, $J(\prod R_i)$ is a left quasi-regular ideal of $\prod R_i$ and so $\{a_i\}$ is a left quasi-regular element of $\prod R_i$. So there is $\{r_i\} \in \prod R_i$ such that $\{r_i\} + \{a_i\} + \{r_i\}\{a_i\} = \{0_i\}$. In particular, $r_k + a_k + r_k a_k = 0_k$ and a_k is left quasi-regular in R_k. Since a_k is an arbitrary element of I_k, then I_k is left quasi-regular in R_k. By Theorem IX.2.3(iv), $I_k \subset J(R_k)$. Since this holds for each $k \in I$, $J(\prod R_i) \subset \prod J(R_i)$. Therefore, $J(\prod R_i) = \prod J(R_i)$, as claimed. □