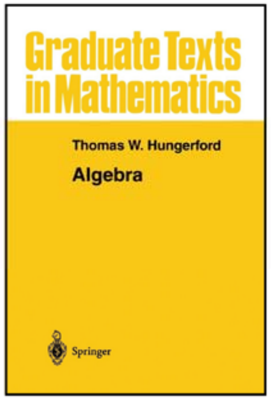
Modern Algebra

Chapter IX. The Structure of Rings

IX.3. Semisimple Rings—Proofs of Theorems



Theorem IX.3.2

Theorem IX.3.2. A nonzero ring R is semisimple if and only if R is isomorphic to a subdirect product or primitive rings.

Proof. Suppose R is nonzero semisimple and let $\mathcal{P}$ be the set of all left primitive ideals of R. Then by Definition IX.2.1, "left primitive ideal," for each $P \in \mathcal{P}$ we have that R/P is a primitive ring. By the definition of semisimple (Definition IX.2.9 which states that R is semisimple if $J(R) = \{0\}$) and Theorem IX.2.3(iii), $\{0\} = J(R) = \cap_{P \in \mathcal{P}} P$. For each $P \in \mathcal{P}$, let $\lambda_P : R \to R/P$ and $\pi_P : \prod_{Q \in \mathcal{P}} R/Q \to R/P$ be the respective canonical epimorphisms. Define $\varphi : R \to \prod_{P \in \mathcal{P}} R/P$ as $r \mapsto \{\lambda_P(r)\}_{P \in \mathcal{P}} = \{r + P\}_{P \in \mathcal{P}}$. If $\varphi(r) = \varphi(s)$ then $\{r + P\}_{P \in \mathcal{P}} = \{s + P\}_{P \in \mathcal{P}}$, or $r + P = s + P$ for all $P \in \mathcal{P}$. So $r - s \in P$ for all $P \in P$, but $\cap_{P \in \mathcal{P}} P = \{0\}$ (this is where the semisimplicity of R is used), so $r - s = 0$ and $r = s$. Therefore φ is one to one and so is a monomorphism.

Theorem IX.3.2 (continued 1)

Proof (continued). Also, for each $r \in R$ we have $\pi_P \varphi(r) = \pi_P(\{r + P\}_{P \in \mathcal{P}}) = r + P$, so $\pi_P \varphi(R) = \{r + P \mid r \in R\} = R/P$. So by Definition IX.3.A, R is isomorphic to a subdirect product of primitive rings.

Conversely, suppose there is a family of primitive rings $\{R_i \mid i \in I\}$ and a monomorphism of rings $\varphi : R \to \prod_{i \in I} R_i$ such that $\pi_k \varphi(R) = R_k$ for each $k \in I$. Let ψ_k be the epimorphism (onto) $\pi_k \varphi$. By Corollary III.2.10 (The First Isomorphism Theorem), $R_k \cong R/\mathrm{Ker}(\psi_k)$ for each $k \in I$ and since R_k is primitive by hypothesis then, by Definition IX.2.1, $\mathrm{Ker}(\psi_k)$ is a left primitive ideal of R. Therefore, by Theorem IX.2.3(ii),

$$J(R) \subset \cap_{k \in I} \mathrm{Ker}(\psi_k). \quad (*)$$

However, if $r \in R$ and $\psi_k(r) = 0 \in R_k$ then the kth component of $\varphi(r)$ in $\prod_{i \in I} R_i$ is zero. Thus if $r \in \cap_{k \in I} \mathrm{Ker}(\psi_k)$, we must have $\varphi(r) = 0$.

Theorem IX.3.2 (continued 2)

Theorem IX.3.2. A nonzero ring R is semisimple if and only if R is isomorphic to a subdirect product or primitive rings.

Proof (continued). Since φ is a monomorphism (one to one) by hypothesis, then we must have $r = 0$ and so $\cap_{k \in I} \mathrm{Ker}(\psi_k) = \{0\}$. This combined with $(*)$ gives $J(R) \subset \cap_{k \in I} \mathrm{Ker}(\psi_k) = \{0\}$. Therefore, $J(R) = \{0\}$ and R is semisimple by Definition IX.2.9, as claimed. □

Theorem IX.3.3

Theorem IX.3.3. The Wedderburn-Artin Theorem for Semisimple Artinian Rings.
The following conditions on a ring R are equivalent.

(i) R is a nonzero semisimple left Artinian ring;

(ii) R is a direct product of a finite number of simple ideals each of which is isomorphic to the endomorphism ring of a finite dimensional vector space over a division ring;

(iii) there exist division rings $D_1, D_2, \dots, D_t$ and $n_1, n_2, \dots, n_t \in \mathbb{N}$ such that R is isomorphic to the ring $\mathrm{Mat}_{n_1}(D_1) \times \mathrm{Mat}_{n_2}(D_2) \times \cdots \times \mathrm{Mat}_{n_t}(D_t)$.

Proof. (ii)⇔(iii) By Theorem VII.1.4, the endomorphism ring of a dimension n vector space over a division ring is isomorphic to $\mathrm{Mat}_n(D)$ for some division ring D. By Exercise III.2.9(a), $\mathrm{Mat}_n(D)$ has no proper ideals so that both it and the isomorphic endomorphism ring are simple. Therefore, (i) and (ii) are equivalent.

Theorem IX.3.3 (continued 1)

Proof (continued). (ii)⇒(i) Suppose R is isomorphic to $\prod_{i=1}^t R_i$ where each R_i is the endomorphism ring of a vector space (and hence R_i is simple by Exercise III.2.9(a), as just explained). It is shown in the example after definition IX.1.5 that each R_i is primitive so, by Theorem 2.10(0), each R_i is semisimple and $J(R_i) = \{0\}$. Consequently, by Theorem IX.2.17,

$$J(R) = J\left(\prod_{i=1}^t R_i\right) \cong \prod_{i=1}^t J(R_i) = \{0\}.$$

So, by definition IX.2.9 of "semisimple," R is semisimple and the first part of (i) holds.

By Theorem VII.1.4, each R_i is isomorphic to $\mathrm{Mat}_n(D)$ for some $n \in \mathbb{N}$ and some division ring D. By Corollary VIII.1.12, $\mathrm{Mat}_n(D)$ (and hence each R_i) is Artinian. By Corollary VIII.1.7, $\prod_{i=1}^t R_i$ (and hence R) is Artinian. So the second claim of (i) holds.

Theorem IX.3.3 (continued 2)

Proof (continued). (i)⇒(ii) Suppose R is a nonzero semisimple Arintian ring. Then by the definition of "semisimple," $J(R) = \{0\}$ and so by theorem IX.2.3(iii) $J(R) = \{0\}$ is the intersection of all left primitive ideals of R, so R has left primitive ideals (or else we would have $J(R) = R \neq \{0\}$; see Note IX.2.A concerning a set theoretic convention which we follow).

Suppose that R has only finitely many distinct left primitive ideals: $P_1, P_2, \dots, P_t$ (we show below that this must be the case). By the definition of "primitive ideal" (Definition IX.2.1), each ring R/P_i is a primitive ring. Now R satisfies the descending chain condition (that is, is Artinian) by hypothesis, so by Corollary VIII.1.6, P_i and R/P_i are Artinian (here we treat R, P_i, and R/P_i as R-modules with $B = R$ in Corollary VIII.1.6). By the Wedderburn-Artin Theorem for Simple Artinian Rings (Theorem IX.1.14, the (ii)⇒(i) and (ii)⇒(iii) parts), each R/P_i is a simple ring isomorphic to an endomorphism ring of a finite dimensional left vector space over a division ring.

Theorem IX.3.3 (continued 3)

Proof (continued). By Theorem III.2.13, since each R/P_i is simple, then each P_i is a maximal ideal of R. Furthermore, $R^2 \not\subset P_i$ (otherwise $(R/P_i)^2 = \{0\}$, which contradicts the definition of "simple ring R/P_i," Definition IX.1.1), whence $P_i \neq R^2 + P_i = R$ (by the maximality of P_i). Likewise, if $i \neq j$ then $P + i \neq P_i + P_j \neq P_j$ and so $P_i + P_j = R$ by maximality. So the hypotheses of Corollary III.2.27 are satisfied and so there is an isomorphism θ mapping $R/(\cap_{i=1}^t P_i) \to R/P_1 \times R/P_2 \times \cdots \times R/P_t$. By Theorem IX.2.3(iii), $J(R) = \cap_{i=1}^t P_i$ and so we have

$$R \cong T/\{0\} = R/J(R) = R\big/\left(\cap_{i=1}^t P_i\right) = R/P_1 \times R/P_2 \times \cdots \times R/P_t.$$

If $\iota_k : R/P_k \to \prod_{i=1}^t R/P_i$ is the canonical injection (see Theorem III.2.22(iv)), then each $\iota_k(R/P_k) = (0_1, 0_2, \dots, 0_{k-1}, R/P_k, 0_{k+1}, \dots, 0_t)$ is a simple ideal of $\prod_{i=1}^r R/P_i$ (since R/P_k is simple).

Theorem IX.3.3 (continued 4)

Proof (continued). Under the isomorphism θ showing $\prod_{i=1}^{t} R/P_i \cong R$, the images of the $\iota_k(R/P_k)$, $\theta(\iota_k(R/P_k))$, are simple ideals of R and

$$R = \theta(\iota_1(R/P_1)) \times \theta(\iota_2(R/P_2)) \times \cdots \times \theta(\iota_t(R/P_t)).$$

We saw above that each R/P_i is isomorphic to an endomorphism ring of a finite dimensional left vector space over a division ring (this is where we used the Wedderburn-Artin Theorem of Simple Artinian Rings), so that each $\theta(\iota_i(R/P_i))$ also satisfies this and hence (ii) holds.

To complete the proof we need only show that R cannot have an infinite number of distinct left primitive ideals. ASSUME that $P_1, P_2, P_3, \ldots$ is a sequence of distinct left primitive ideals of R. An intersection of (left) ideals is again a (left) ideal by Corollary III.2.3, so $P_1 \supset P_1 \cap P_2 \supset P_1 \cap P_2 \cap P_3 \supset \cdots$ is a descending chain of (left) ideals of R.

Theorem IX.3.3 (continued 5)

Proof (continued). Since R is Artinian by hypothesis, there is $n \in \mathbb{N}$ such that $P_1 \cap P_2 \cap \cdots \cap P_n = P_1 \cap P_2 \cap \cdots \cap P_n \cap P_{n+1}$ (in fact, all intersections beyond this point must be equal), whence $P_1 \cap P_2 \cap \cdots \cap P_n \subset P_{n+1}$. We saw above that $R^2 + P_1 = R$ and $P_1 + P_j = R$ (for $i \neq j$) for $i, j = 1, 2, \ldots, n+1$. The proof of Theorem III.2.25 (see line 5 of page 132) shows that $P_{n+1} + (P_1 \cap P_2 \cap \cdots \cap P_n) = R$. Consequently $P_{n+1} = R$. But P_{n+1} is a left primitive ideal of R, and R itself is not a left primitive ideal of R (see the Note after Definition IX.2.1 of "left primitive ideal"), a CONTRADICTION. So the assumption that R has infinitely many distinct left primitive ideals is false, and the proof is complete. □

Corollary IX.3.4

Corollary IX.3.4.

(i) A semisimple left Artinian ring has an identity.

(ii) A semisimple ring is left Artinian if and only if it is right Artinian.

(iii) A semisimple left Artinian ring is both left and right Noetherian.

Proof. (i) By Theorem IX.3.3 (the (i)⇒(iii) part), R is isomorphic to $\text{Mat}_{n_1}(D_1) \times \text{Mat}_{n_2}(D_2) \times \cdots \times \text{Mat}_{n_t}(D_t)$ for dome $n_1, n_2, \ldots, n_t \in \mathbb{N}$ and for some division rings $D_1, D_2, \ldots, D_t$. Since a division ring contains an identity, then each $\text{Mat}_{n_i}(D_i)$ contains an identity (the usual $n_i \times n_i$ identity matrix) and so the direct product and hence R has an identity.

(ii) Theorem IX.3.3 holds if "left" is replaced with "right." If R is a semisimple left Artinian ring then by Theorem IX.3.3 (the (i)⇒(iii) part), R is isomorphic to $\text{Mat}_{n_1}(D_1) \times \text{Mat}_{n_2}(D_2) \times \cdots \times \text{Mat}_{n_t}(D_t)$ for *some* $t, n_1, n_2, \ldots, n_t \in \mathbb{N}$ and some division rings $D_1, D_2, \ldots, D_t$.

Corollary IX.3.4 (continued)

Proof (continued). Now by Theorem IX.3.3 (the (iii)⇒(i) part with "left" replaced with "right"), R is a semisimple right Artinian ring, as claimed.

(iii) Let A be a semisimple left Artinian ring. By Theorem IX.3.3 (the (i)⇒(iii) part),

$$R \cong \text{Mat}_{n_1}(D_1) \times \text{Mat}_{n_2}(D_2) \times \cdots \times \text{Mat}_{n_t}(D_t)$$

for some $t, n_1, n_2, \ldots, n_t \in \mathbb{N}$ and for some division rings $D_1, D_2, \ldots, D_t$. By Corollary VIII.1.12, each $\text{Mat}_{n_i}(D_i)$ is Noetherian. By Corollary VIII.1.7, $\text{Mat}_{n_1}(D_1) \times \text{Mat}_{n_2}(D_2) \times \cdots \times \text{Mat}_{n_t}(D_t)$ (and hence R) is then Noetherian, as claimed. □

Corollary IX.3.5

Corollary IX.3.5. If I is an ideal in a semisimple left Artinian ring R, then $I = Re$, where e is an idempotent element (that is, $e^2 = e$) which is in the center of R.

Proof. Let R be a left Artinian semisimple ring. By Theorem IX.3.3(ii), R is a (ring) direct product of simple ideals, say $R = I_1 \times I_2 \times \cdots \times I_n$. Since each I_j is simple, then for a given ideal I of R we have $I \cap I_j$ is either $\{0\}$ or I_j (since the intersection of ideals is an ideal by Corollary III.2.3). This also holds for $I = I_i$, so R is an internal direct product of the I_j and we do not treat R as a collection of n-tuples but instead note that each element of R is a unique sum of elements of the I_j (the uniqueness follows from Theorem I.8.9); see the Note after Theorem III.2.24 (and page 131 of Hungerford). After re-indexing (if necessary) we may assume that $I \cap I_j = I_j$ for $j = 1, 2, \ldots, t$ and $I \cap I_j = \{0\}$ for $j = t+1, t+2, \ldots, n$. By Corollary IX.2.4(i), R has an identity 1_R. So $1_R = e_1 + e_2 + \cdots + e_n$ for some $e_i \in I_i$.

Corollary IX.3.5 (continued)

Proof (continued). Now $I_i I_j \subset I_k$ since I_k is a left ideal of R and $I_j \subset R$, and $I_j I_k \subset I_j$ since I_j is a right ideal of R and $I_k \subset R$. So $I_j I_k \subset I_j \cap I_k = \{0\}$ for $j \neq k$. Therefore,

$$e_1 + e_2 + \cdots + e_n = 1_R = (1_r)^2 = (e_1 + e_2 + \cdots + e_n)^2 = e_1^2 + e_2^2 + \cdots + e_n^2$$

and so $e_j = e_j^2$ for each j. Similarly, $(e_1 + e_2 + \cdots + e_t)^2 = e_1 + e_2 + \cdots + e_t$ and $(e_{t+1} + e_{t+2} + \cdots + e_n)^2 = e_{t+1} + e_{t+2} + \cdots + e_n$. So that $e_1 + e_2 + \cdots + e_t$ and $e_{t+1} + e_{t+2} + \cdots + e_n$ are idempotent. With $e = e_1 + e_2 + \cdots + e_t$ we have by Exercise III.2.23 that $1_R - (e_{t+1} + e_{t+2} + \cdots + e_t) = e_1 + e_2 + \cdots + e_t = e$ is in the center of R. Since I is an ideal, $Re \subset I$. Conversely, if $u \in I$ then $u = u1_R = ue_1 + ue_2 + \cdots + ue_n$, but for $j = t+1, t+2, \ldots, n$ we have $ue_j \in I \cap I_j = \{0\}$ and thus $u = ue_1 + ue_2 + \cdots + ue_t = ue$. So $ue \in I$ and $I \subset Re$. Hence $I = Re$ for idempotent e in the center of R, as claimed. □