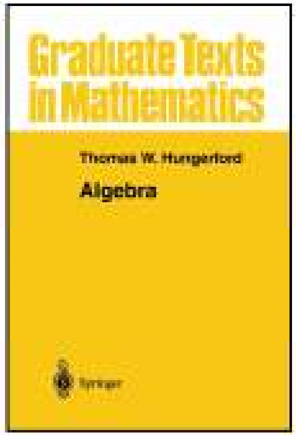
Modern Algebra

Chapter V. Fields and Galois Theory

V.1.Appendix. Ruler and Compass Constructions—Proofs of Theorems



Lemma V.1.15

Lemma V.1.15. Let F be a subfield of the field $\mathbb{R}$ of real numbers and let L_1, L_2 be nonparallel lines in F and C_1, C_2 distinct circles in F. Then

(i) $L_1 \cap L_2$ is a point in the plane of F;

(ii) $L_1 \cap C_1 = \varnothing$ or consists of one or two points in the plane of $F(\sqrt{u})$ for some $u \in F$ where $u \geq 0$;

(iii) $C_1 \cap C_2 = \varnothing$ or consists of one or two points in the plane of $F(\sqrt{u})$ for some $u \in F$ where $u \geq 0$.

Proof. (i) Let L_1 have equation $a_1x + b_1y + c_1 = 0$ and let line L_2 have equation $a_2x + b_2y + c_2 = 0$. Then we find that the only common point to L_1 and L_2 is $x = (b_1c_2 - b_2c_1)/(a_1b_2 - a_2b_1)$ and $y =)a_1c_2 - a_2c_1)/(a_2b_1 - a_1b_2)$ where $a_1b_2 - a_2b_1 \neq 0$ since L_1 and L_2 are nonparallel. Notice that $x, y \in F$ since F is a field.

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(iii) $C_1 \cap C_2 = \varnothing$ or consists of one or two points in the plane of $F(\sqrt{u})$ for some $u \in F$ where $u \geq 0$.

Proof. (iii) Let C_1 have equation $x^2 + y^2 + a_1x + b_1y + c_1 = 0$ and let C_2 have equation $x^2 + y^2 + a_2x + b_2y + c_2 = 0$ where $a_1, a_2, b_1, b_2, c_1, c_2 \in F$. Then if (x, y) lies on both C_1 and C_2, we also have that (x, y) lies on the line L with equation $(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2) = 0$ (from "subtracting C_2 from C_1"). So a point (x, y) lies on both C_1 and C_2 if and only if it lies on both C_1 and L. So case (iii) reduces to case (ii).

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(ii) $L_1 \cap C_1 = \varnothing$ or consists of one or two points in the plane of $F(\sqrt{u})$ for some $u \in F$ where $u \geq 0$.

Proof. (ii) Suppose line L_1 has the equation $dx + ey + f = 0$ where $d, e, f \in F$ (and C_1 has the equation given above).
If $d = 0$ then $e \neq 0$ and the only (x, y) on both L and C_1 satisfies $ey + f = 0$ and $x^2 + y^2 + a_1x + b_1y + c_1 = 0$. Then we have $y = -f/e$ and so $x^2 + (-f/e)^2 + a_1x + b_1(-f/e) + c_1 = 0$ or $e^2x^2 + e^2a_1x + (f^2 - efb_1 + e^2c_1) = 0$. The quadratic equation then gives

$$x = \frac{-e^2a_1 \pm \sqrt{(e^2a_1)^2 - 4(e^2)(f^2 - efb_1 + e^2c_1)}}{2e^2}.$$

Let $u = (e^2a_1)^2 - 4(e^2)(f^2 - efb_1 + e^2c_1)$. If $u < 0$ then $L_1 \cap C_1 = \varnothing$.

Lemma V.1.15 (continued)

Proof (continued). (ii) If $u = 0$ then $x = -a_1/2$ and $y = -f/e$ and there is one point on both L_1 and C_1. If $u > 0$ then there are two points on L_1 and C_1 and x is in terms of $\sqrt{u}$; so the two points lie in $F(\sqrt{u})$. If $d \neq 0$ then we can "normalize" the equation for L_1 and WLOG assume $d = 1$, so that $x + ey + f = 0$, or $x = -ey - f$. So a point (x, y) lies on both L_1 and C_1 then
$(-ey - f)^2 + y^2 + a_1(-ey - f) + b_1 y + c_1 = Ay^2 + By + C = 0$ where $A, B, C \in F$. If $A = 0$ then $y \in F$ and so $x \in F$. Then $x, y \in F = F(\sqrt{1})$. If $A \neq 0$ then again by normalizing we may assume $A = 1$ and we need $y^2 + By + C = 0$. Completing the square yields
$(y + B/2)^2 + (C - B^2/4) = 0$. This gives $y = -B/2 \pm \sqrt{-C + B^2/4}$. Let $u = -C + B^2/4$. Then $L_1 \cap C_1 = \varnothing$ if $u < 0$. If $u = 0$ then there is one point (x, y) on $L_1 \cap C_1$ where $x, y \in F = F(0)$. If $u > 0$ then there are two points (x, y) on $L_1 \cap C_1$ both of which satisfy $x, y \in F(\sqrt{u})$. □

Proposition V.1.16

Proposition V.1.16. If a real number c is constructible, then c is algebraic of degree a power of 2 over the field $\mathbb{Q}$ or rationals.

Proof. From the fact that every integer is constructible, along with the previous note, shows that $\mathbb{Q}$ consists of constructible numbers and so we take the plane of $\mathbb{Q}$ as given. The only way to construct new points is to find the intersection of lines and/or circles. Now to construct a line or circle we need two points (the center P and radius PT for a circle). The two points must either lie in the plane of $\mathbb{Q}$ or be points previously constructed through a finite sequence of intersections of lines and/or circles.

Let c be constructible. Then c results from a finite sequence of intersections of constructible lines and/or circles (starting with the plane or $\mathbb{Q}$). By Lemma V.1.15, the first point so constructed lies in the plane of an extension field $\mathbb{Q}(\sqrt{u})$ of $\mathbb{Q}$ with $u \in \mathbb{Q}$, or equivalently in the plane of $\mathbb{Q}(v)$ with $v^2 \in \mathbb{Q}$. Such an extension has degree 1 or degree 2 over $\mathbb{Q}$.

Proposition V.1.16 (continued)

Proposition V.1.16. If a real number c is constructible, then c is algebraic of degree a power of 2 over the field $\mathbb{Q}$ or rationals.

Proof (continued). Similarly, the next new point constructed lies in the plane of $Q(v, w)$ with $w^2 \in \mathbb{Q}(v)$ (again, by Lemma V.1.15). So $(c, 0)$ lies in the plane of $F = \mathbb{Q}(v_1, v_2, \ldots, v_n)$ for some $n \in \mathbb{N}$ where $\mathbb{Q} \subset \mathbb{Q}(v_1) \subset \mathbb{Q}(v_1, v_2) \subset \cdots \subset \mathbb{Q}(v_1, v_2, \ldots, v_n)$ with $v_i^2 \in \mathbb{Q}(v_1, v_2, \ldots, v_{i-1})$ and by Lemma V.1.15, $[\mathbb{Q}(v_1, v_2, \ldots, v_i) : \mathbb{Q}(v_1, v_2, \ldots, v_{i-1})] \in \{1, 2\}$ for $2 \leq i \leq n$. By Theorem V.1.2, $[F : \mathbb{Q}]$ is the product of these dimensions and so $[F : \mathbb{Q}]$ is a power of two. So by Theorem V.1.11, c is algebraic over $\mathbb{Q}$. Now (as fields) $\mathbb{Q} \subset \mathbb{Q}(c) \subset F$ and so by Theorem V.1.2, $[\mathbb{Q}(c) : \mathbb{Q}][F : \mathbb{Q}(c)] = [F : \mathbb{Q}]$ and so $[\mathbb{Q}(c) : \mathbb{Q}]$ divides $[F : \mathbb{Q}]$. So the degree $[\mathbb{Q}(c) : \mathbb{Q}]$ of c over $\mathbb{Q}$ is a power of 2. □

Corollary V.1.17

Corollary V.1.17. Straight Edge and Compass Trisection of a General Angle is Impossible.
An angle of 60° cannot be trisected by ruler and compass constructions, and therefore a general angle cannot be trisected.

Proof. If it were possible to trisect a 60° angle, we would then be able to construct a right triangle with one acute angle of 20°. It would then be possible to construct the real number $\cos(20^\circ)$ (see Exercise V.1.25(b) or the the Lemma to Theorem 32.11 in my YouTube video online at `https://www.youtube.com/watch?v=S24GYj1rWGs`, accessed 12/20/2015). However for any angle α, elementary trigonometric shows that $\cos(3\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha)$. With $\alpha = 20^\circ$, then $\cos(3\alpha) = \cos(60^\circ) = 1/2$ and $\cos(20^\circ)$ is a root of the polynomial equation $1/2 = 4x^3 - 3x$ or equivalently $8x^3 - 6x - 1 = 0$.

Corollary V.1.17

Corollary V.1.17. Straight Edge and Compass Trisection of a General Angle is Impossible.
An angle of $60°$ cannot be trisected by ruler and compass constructions, and therefore a general angle cannot be trisected.

Proof. But $8x^3 - 6x - 1$ is irreducible in $\mathbb{Q}[x]$ by Proposition III.6.8 and the Factor Theorem (Theorem III.6.6). Therefore, $\cos(20°)$ has degree 3 over $\mathbb{Q}$ and so $\cos(20°)$ is not constructible by Proposition V.1.16, and whence a $20°$ angle is not constructible. □

Corollary V.1.18

Corollary V.1.18. Straight Edge and Compass Doubling of the Cube is Impossible.
It is impossible by ruler and compass constructions to duplicate a cube of side length 1 (that is, to construct the side of a cube of volume 2).

Proof. If s is the side length of a cube of volume 2, then s is a root of $x^3 - 2$ which is irreducible in $\mathbb{Q}[x]$ by Eisentein's Criterion (Theorem III.6.15). Therefore x is not constructible by Proposition V.1.16 since $\sqrt[3]{2}$ is of degree 3 over $\mathbb{Q}$. □

Corollary V.1.19

Corollary V.1.19. Straight Edge and Compass Squaring of the Circle is Impossible.
It is impossible by ruler and compass constructions to construct a square with area equal to the area of a circle of radius 1 (that is, to construct a square with area π).

Proof. Consider a circle of radius 1, and so area π. ASSUME a square of area π can be constructed. Then the length of a side of the square is $\sqrt{\pi}$ and this is a constructible number. Then π is constructible and so by Proposition V.1.16, π is algebraic over $\mathbb{Q}$. But π is known to be transcendental by Lindemann's proof, a CONTRADICTION. So no such square is constructible. □