Chapter V. Fields and Galois Theory
V.2. The Fundamental Theorem (of Galois Theory)—Proofs of Theorems
Theorem V.2.2. Let $F$ be an extension field of $K$ and $K[x]$. If $u \in F$ is a root of $f$ and $\sigma \in \text{Aut}_K(F)$, then $\sigma(u) \in F$ is also a root of $f$.

**Proof.** Let $f = \sum_{i=0}^{n} k_i x^i$. Since $\sigma$ fixes $K$, $\sigma(0) = 0$ and so $f(u) = 0$ implies

$$0 = \sigma(0) = \sigma(f(u)) = \sigma \left( \sum_{i=0}^{n} k_i x^i \right)$$

$$= \sum_{i=0}^{n} \sigma(k_i) \sigma(u^i) = \sum_{i=0}^{n} k_i (\sigma(u))^i = f(\sigma(u)).$$
Theorem V.2.2. Let $F$ be an extension field of $K$ and $K[x]$. If $u \in F$ is a root of $f$ and $\sigma \in \text{Aut}_K(F)$, then $\sigma(u) \in F$ is also a root of $f$.

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Lemma V.2.6

Lemma V.2.6. Let $F$ be an extension field of $K$ with intermediate fields $L$ and $M$ (say $K \subset L \subset M \subset F$). Let $H$ and $J$ be subgroups of $G = \text{Aut}_K(F)$. Then:

(i) $F' = 1$ (the identity group) and $K' = G$;

(i') $1' = F$;

(ii) $L \subset M$ implies $M' < L'$;

(ii') $H < J$ implies $J' \subset H'$;

(iii) $L \subset L''$ and $H < H''$ (where $L'' = (L')'$ and $H'' = (H')'$);

(iv) $L' = L'''$ and $H' = H'''$.

Proof. (i) Now $F' = \text{Aut}_F(F)$ is the group of automorphisms of $F$ which fix $F$ and hence must consist only of the identity permutation and so $F'$ is the “identity group.”
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(i') $1'$ is the fixed field of the identity group. $F$ is the “universal field” and the identity group fixes all of $F$; i.e., $1' = F$. 
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Lemma V.2.6 (ii)

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(ii) $L \subset M$ implies $M' < L'$;

(ii') $H < J$ implies $J' \subset H'$.

Proof. (ii) Suppose the intermediate fields $L, M$ satisfy $L \subset M$. 

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**Proof.** (ii) Suppose the intermediate fields $L, M$ satisfy $L \subset M$. An element of $M' = \text{Aut}_M(F)$ fixes $M$ and with $L \subset M$ such an element must also fix $L$ and so the element is in $L' = \text{Aut}_L(F)$. So $M' \subset L'$. 
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(ii') Suppose subgroups $H, J$ of $G = \text{Aut}_K(F)$ satisfy $H < J$. Now an element of $J'$ (the fixed field of $J$) is fixed by every element of $J$ and, since $H < J$, also fixed by every element of $H$.
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Proof. (iii) Let $L$ be an intermediate field. Then $L' = \text{Aut}_L(F)$ is a group, and $L''$ is the fixed field of $L'$. Now any element of $L$ is fixed by $L' = \text{Aut}_L(F)$. 
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Proof. (iii) Let $L$ be an intermediate field. Then $L' = \text{Aut}_L(F)$ is a group, and $L''$ is the fixed field of $L'$. Now any element of $L$ is fixed by $L' = \text{Aut}_L(F)$. Also, $L''$ includes everything in $F$ fixed by the elements of $L' = \text{Aut}_L(F)$. So $L''$ includes all of $L$ (and possibly more); $L \subset L''$. 
Lemma V.2.6 (iii)

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Let $H$ by a subgroup of $G = \text{Aut}_K(F)$. Then $H'$ is the fixed field of $H$. Now $(H')' = H''$ is the group of permutations of $F$ which fix $H'$. 

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(iii) $L \subset L''$ and $H < H''$ (where $L'' = (L')'$ and $H'' = (H')'$).

**Proof.** (iii) Let $L$ be an intermediate field. Then $L' = \text{Aut}_L(F)$ is a group, and $L''$ is the fixed field of $L'$. Now any element of $L$ is fixed by $L' = \text{Aut}_L(F)$. Also, $L''$ includes everything in $F$ fixed by the elements of $L' = \text{Aut}_L(F)$. So $L''$ includes all of $L$ (and possibly more); $L \subset L''$.

Let $H$ by a subgroup of $G = \text{Aut}_K(F)$. Then $H'$ is the fixed field of $H$. Now $(H')' = H''$ is the group of permutations of $F$ which fix $H'$. So every element of $H$ fixes all of $H'$ and such an element is therefore also in $H''$. So $H < H''$. 
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Lemma V.2.6. Let $F$ be an extension field of $K$ with intermediate fields $L$ and $M$ (say $K \subset L \subset M \subset F$). Let $H$ and $J$ be subgroups of $G = \text{Aut}_K(F)$. Then:

(iv) $L' = L'''$ and $H' = H'''$.

Proof. (iv) Let $L$ be an intermediate field. By (iii), $L \subset L''$ and so by (ii), $L''' < L'$. Now $L'$ is a subgroup of $G = \text{Aut}_K(F)$ and so by (iii) (with $H$ replaced with $L'$) we have $L' < L'''$, and so $L' = L'''$.

Let $H$ be a subgroup of $G = \text{Aut}_K(F)$. By (iii), $H < H''$ and so by (ii'), $H''' \subset H'$. Now $H'$ is an intermediate field so by (iii) (with $L$ replaced with $H'$) we have $H' \subset H'''$, and so $H' = H'''$. \qed
Lemma V.2.6. Let $F$ be an extension field of $K$ with intermediate fields $L$ and $M$ (say $K \subset L \subset M \subset F$). Let $H$ and $J$ be subgroups of $G = \text{Aut}_K(F)$. Then:

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Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

Proof. (Notice that $[M : L]$ and $[F : K]$ are dimensions of vector spaces; $[L' : M']$, the index of $L'$ over $M'$, is the number of cosets of $L$ in $M$.)
Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subseteq M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

Proof. (Notice that $[M : L]$ and $[F : K]$ are dimensions of vector spaces; $[L' : M']$, the index of $L'$ over $M'$, is the number of cosets of $L$ in $M$.) Since $[M : L]$ is finite, we give a proof based on induction. Let $n = [M : L]$. If $n = 1$ then $M + L$ and so $M' = L'$ and $[L' : M'] = 1$, so the result holds.
Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

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Let $n > 1$ and suppose the theorem holds for all $i < n$. Since $n > 1$, there is some $u \in M$ with $u \not\in L$. Since $[M : L]$ is finite, then $u$ is algebraic over $L$ by Theorem V.1.11. Let $f \in L[x]$ be the irreducible monic polynomial of $u$, say of degree $k > 1$. 
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Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subseteq M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

Proof. (Notice that $[M : L]$ and $[F : K]$ are dimensions of vector spaces; $[L' : M']$, the index of $L'$ over $M'$, is the number of cosets of $L$ in $M$.) Since $[M : L]$ is finite, we give a proof based on induction. Let $n = [M : L]$. If $n = 1$ then $M + L$ and so $M' = L'$ and $[L' : M'] = 1$, so the result holds. Let $n > 1$ and suppose the theorem holds for all $i < n$. Since $n > 1$, there is some $u \in M$ with $u \not\in L$. Since $[M : L]$ is finite, then $u$ is algebraic over $L$ by Theorem V.1.11. Let $f \in L[x]$ be the irreducible monic polynomial of $u$, say of degree $k > 1$. By Theorem V.1.6(iii), $[L(u) : L] = k$. By Theorem V.1.1, $[M : L] = [M : L(u)][L(u) : L]$ and so $[M : L(u)] = n/k$. 
Lemma V.2.8 (continued 1)

Proof (continued). Schematically:

\[
\begin{array}{ccc}
\frac{n}{k} & \rightarrow & M' \\
\cup & \Lambda & \\
L(u) & \rightarrow & (L(u))' \\
\cup & \Lambda & \\
L & \rightarrow & L'
\end{array}
\]

We now consider two cases.

Case 1. If \(k < n\) then \(1 < n/k < n\); and

Case 2. If \(k = n\).
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**Case 1.** If $k < n$ then $1 < n/k < n$; and

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Proof (continued).
Case 1. If $k < n$ then $1 < n/k < n$. By the induction hypothesis, since $i = n/k < n$, we have that $L \subset L(u)$ implies $[L' : (L(u))'] \leq [L(u) : L] = k$, and that $L(U) \subset M$ implies $[(L(u))' : M'] \leq [M : L(u)] = n/k$. 
Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

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$$[L' : M'] = [L' : (L(u))'][(L(u))' : M'] \leq k(n/k) = n = [M : L]$$

and the theorem holds in this case.
Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

Proof (continued).

Case 1. If $k < n$ then $1 < n/k < n$. By the induction hypothesis, since $i = n/k < n$, we have that $L \subset L(u)$ implies $[L' : (L(u))'] \leq [L(u) : L] = k$, and that $L(U) \subset M$ implies $[(L(u))' : M'] \leq [M : L(u)] = n/k$. Hence

$$[L' : M'] = [L' : (L(u))'][(L(u))' : M'] \text{ by Theorem V.1.1}$$

$$\leq k(n/k) = n = [M : L]$$

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Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subseteq M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

Proof (continued).
Case 2. On the other hand, if $k = n$ then by Theorem V.1.1, $[M : L] = [M : L(u)][L(u) : L]$ and so $[M'' L(u)] = 1$ (as above). So $M = L(u)$. 
Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

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Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

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Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

Proof (continued).
Case 2. On the other hand, if $k = n$ then by Theorem V.1.1, $[M : L] = [M : L(u)][L(u) : L]$ and so $[M'' L(u)] = 1$ (as above). So $M = L(u)$. In the final part of the proof, we will construct an injective map from the set $S$ of all left costs of $M'$ in $L'$ (of which there are $[L' : M']$ such cosets) to the set $T$ of all distinct roots in $F$ of the polynomial $f \in L[x]$ (of which there are at most $k \leq n$ such roots by Theorem III.6.7). So we have $|S| = [L' : M']$ and $|T| \leq n$, the existence of the injective map from $S$ to $T$ gives that $|S| \leq |T|$ and it will then follow that $[L' : M'] = |S| \leq |T| \leq n = [M : L]$, establishing the theorem in this second case.
Proof (continued). Now for the construction of the injective map from $S$ to $T$. Let $\tau \in L'$ and $\tau M'$ a left coset of $M'$ in $L'$. If $\sigma \in M' = \text{Aut}_M(F)$, then since $u \in M$ (by choice, above) we have that $\sigma(u) = u$ and so $\tau \sigma(u) = \tau(u)$; so every element of the coset $\tau M'$ (this is a group element which acts on elements of $F$, $u$ in particular) has the same effect on $u$ and maps $u \mapsto \tau(u)$ (that is, there is independence of element $\sigma \in M'$).
Lemma V.2.8 (continued 4)

**Proof (continued).** Now for the construction of the injective map from $S$ to $T$. Let $\tau \in L'$ and $\tau M'$ a left coset of $M'$ in $L'$. If $\sigma \in M' = \text{Aut}_M(F)$, then since $u \in M$ (by choice, above) we have that $\sigma(u) = u$ and so $\tau \sigma(u) = \tau(u)$; so every element of the coset $\tau M'$ (this is a group element which acts on elements of $F$, $u$ in particular) has the same effect on $u$ and maps $u \mapsto \tau(u)$ (that is, there is independence of element $\sigma \in M'$). Since $\tau \in L' = \text{Aut}_L(F)$ (because $\tau M'$ is a coset in $L'$) and $u$ is a root of $f \in L[x]$, then $\tau(u)$ is also a root of $f$ by Theorem V.2.2. This implies that the map $S \mapsto T$ given by $\tau M' \mapsto \tau(u)$ is well-defined (HMMMMM; that is, the mapping actually produces an element of $T$, the set of roots of $f$).
Lemma V.2.8 (continued 4)

**Proof (continued).** Now for the construction of the injective map from $S$ to $T$. Let $\tau \in L'$ and $\tau M'$ a left coset of $M'$ in $L'$. If $\sigma \in M' = \text{Aut}_M(F)$, then since $u \in M$ (by choice, above) we have that $\sigma(u) = u$ and so $\tau \sigma(u) = \tau(u)$; so every element of the coset $\tau M'$ (this is a group element which acts on elements of $F$, $u$ in particular) has the same effect on $u$ and maps $u \mapsto \tau(u)$ (that is, there is independence of element $\sigma \in M'$). Since $\tau \in L' = \text{Aut}_L(F)$ (because $\tau M'$ is a coset in $L'$) and $u$ is a root of $f \in L[x]$, then $\tau(u)$ is also a root of $f$ by Theorem V.2.2. This implies that the map $S \mapsto T$ given by $\tau M' \mapsto \tau(u)$ is well-defined (HMMMM; that is, the mapping actually produces an element of $T$, the set of roots of $f$). If $\tau(u) = \tau_0(u)$ for $\tau, \tau_0 \in L'$ then $\tau_0^{-1} \tau(u) = u$ ($L'$ is a group of permutations, so inverses exist) and hence $\tau_0 \tau$ fixes $u$. Since $\tau, \tau_0 \in L' = \text{Aut}_L(F)$ then certainly $\tau, \tau_0$, and $\tau_0^{-1}$ fixes $L$, so $\tau_0^{-1} \tau$ fixes $L(u) = M$ elementwise (recall that a basis for $L(u) = M$ over $L$ is \{1, u, u^2, \ldots, u^{k-1}\} by Theorem V.1.6(iv)) and $\tau_0 \tau \in M'$. 
Lemma V.2.8 (continued 4)

Proof (continued). Now for the construction of the injective map from $S$ to $T$. Let $\tau \in L'$ and $\tau M'$ a left coset of $M'$ in $L'$. If $\sigma \in M' = \text{Aut}_M(F)$, then since $u \in M$ (by choice, above) we have that $\sigma(u) = u$ and so $\tau \sigma(u) = \tau(u)$; so every element of the coset $\tau M'$ (this is a group element which acts on elements of $F$, $u$ in particular) has the same effect on $u$ and maps $u \mapsto \tau(u)$ (that is, there is independence of element $\sigma \in M'$). Since $\tau \in L' = \text{Aut}_L(F)$ (because $\tau M'$ is a coset in $L'$) and $u$ is a root of $f \in L[x]$, then $\tau(u)$ is also a root of $f$ by Theorem V.2.2. This implies that the map $S \mapsto T$ given by $\tau M' \mapsto \tau(u)$ is well-defined (HMMMM; that is, the mapping actually produces an element of $T$, the set of roots of $f$). If $\tau(u) = \tau_0(u)$ for $\tau, \tau_0 \in L'$ then $\tau_0^{-1} \tau(u) = u$ ($L'$ is a group of permutations, so inverses exist) and hence $\tau_0 \tau$ fixes $u$. Since $\tau, \tau_0 \in L' = \text{Aut}_L(F)$ then certainly $\tau, \tau_0$, and $\tau_0^{-1}$ fixes $L$, so $\tau_0^{-1} \tau$ fixes $L(u) = M$ elementwise (recall that a basis for $L(u) = M$ over $L$ is $\{1, u, u^2, \ldots, u^{k-1}\}$ by Theorem V.1.6(iv)) and $\tau_0 \tau \in M'$. 
Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

Proof (continued). Consequently by Corollary V.4.3(iii), $\tau_0 M' = \tau M'$ and so the map $S \to T$ is one to one (injective) and this completes the second case of the induction. Therefore $[L' : M'] \leq [M : L]$.

For the “in particular” part of the proof, notice that $\text{Aut}_K(F) \cong \text{Aut}_K(F)/1$ (where “1” is the trivial “identity group”). So $|\text{Aut}_K(F)| = [\text{Aut}_K(F) : 1]$. 
Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

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For the “in particular” part of the proof, notice that $\text{Aut}_K(F) \cong \text{Aut}_K(F)/1$ (where “1” is the trivial “identity group”). So $|\text{Aut}_K(F)| = [\text{Aut}_K(F) : 1]$. Also, in the prime notation $K' = \text{Aut}_K(F)$ and $F' = \text{Aut}_F(F) = 1$, so $|\text{Aut}_K(F)| = [\text{Aut}_K(F) : 1] = [K' : F'] \leq [F : K]$ with $L = K$ and $M = F$, from the above result.
Lemma V.2.8. Let $F$ be an extension field of $K$ and $L, M$ intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

Proof (continued). Consequently by Corollary V.4.3(iii), $\tau_0 M' = \tau M'$ and so the map $S \to T$ is one to one (injective) and this completes the second case of the induction. Therefore $[L' : M'] \leq [M : L]$.

For the “in particular” part of the proof, notice that $\text{Aut}_K(F) \cong \text{Aut}_K(F)/1$ (where “1” is the trivial “identity group”). So $|\text{Aut}_K(F)| = [\text{Aut}_K(F) : 1]$. Also, in the prime notation $K' = \text{Aut}_K(F)$ and $F' = \text{Aut}_F(F) = 1$, so $|\text{Aut}_K(F)| = [\text{Aut}_K(F) : 1] = [K' : F'] \leq [F : K]$ with $L = K$ and $M = F$, from the above result. \qed
Lemma V.2.9

**Lemma V.2.9.** Let $F$ be an extension field of $K$ and let $H, J$ be subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$. If $[J : H]$ is finite, then $[H' : J'] \leq [J : H]$.

**Proof.** (Notice that $[H' : J']$ is the dimension of field $H'$ as a vector space over field $J'$; the index $[J : H]$ is the number of cosets of $H$ in $J$.) Let the number of cosets of $H$ in $J$ by $[J : H] = n$. 
Lemma V.2.9. Let $F$ be an extension field of $K$ and let $H, J$ be subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$. If $[J : H]$ is finite, then $[H' : J'] \leq [J : H]$.

Proof. (Notice that $[H' : J']$ is the dimension of field $H'$ as a vector space over field $J'$; the index $[J : H]$ is the number of cosets of $H$ in $J$.) Let the number of cosets of $H$ in $J$ by $[J : H] = n$. Assume $[H' : J'] > n$. Then a basis of $H'$ over $J'$ has more than $n$ elements (as basis is a linearly independent spanning set; see page 181) and so there exist $u_1, u_2, \ldots, u_{n+1} \in H'$ that are linearly independent over $J'$. 
Lemma V.2.9. Let $F$ be an extension field of $K$ and let $H, J$ be subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$. If $[J : H]$ is finite, then $[H' : J'] \leq [J : H]$.

Proof. (Notice that $[H' : J']$ is the dimension of field $H'$ as a vector space over field $J'$; the index $[J : H]$ is the number of cosets of $H$ in $J$.) Let the number of cosets of $H$ in $J$ by $[J : H] = n$. ASSUME $[H' : J'] > n$. Then a basis of $H'$ over $J'$ has more than $n$ elements (as basis is a linearly independent spanning set; see page 181) and so there exist $u_1, u_2, \ldots, u_{n+1} \in H'$ that are linearly independent over $J'$. Let $\{\tau_1, \tau_2, \ldots, \tau_n\}$ be a complete set of representatives of the $n$ left cosets of $H$ in $J$. That is, $J = \tau_1 H \cup \tau_2 H \cup \cdots \cup \tau_n H$ (since cosets of a subgroup partition the group; Corollary I.4.3(i),(ii)) and $\tau_i^{-1} \tau_j \in H$ if and only if $i = j$ by Corollary I.4.3(iii). Consider the system of $n$ homogeneous linear equations in $n + 1$ unknowns with coefficients $\tau_i(u_j)$ in field $F$:
Lemma V.2.9. Let $F$ be an extension field of $K$ and let $H, J$ be subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$. If $[J : H]$ is finite, then $[H' : J'] \leq [J : H]$.

Proof. (Notice that $[H' : J']$ is the dimension of field $H'$ as a vector space over field $J'$; the index $[J : H]$ is the number of cosets of $H$ in $J$.) Let the number of cosets of $H$ in $J$ by $[J : H] = n$. Assume $[H' : J'] > n$. Then a basis of $H'$ over $J'$ has more than $n$ elements (as basis is a linearly independent spanning set; see page 181) and so there exist $u_1, u_2, \ldots, u_{n+1} \in H'$ that are linearly independent over $J'$. Let 

$\{\tau_1, \tau_2, \ldots, \tau_n\}$ be a complete set of representatives of the $n$ left cosets of $H$ in $J$. That is, $J = \tau_1H \cup \tau_2H \cup \cdots \cup \tau_nH$ (since cosets of a subgroup partition the group; Corollary I.4.3(i),(ii)) and $\tau_i^{-1}\tau_j \in H$ if and only if $i = j$ by Corollary I.4.3(iii). Consider the system of $n$ homogeneous linear equations in $n + 1$ unknowns with coefficients $\tau_i(u_j)$ in field $F$: 

(continued on next page)
Lemma V.2.9 (continued 1)

\[ \begin{align*}
\tau_1(u_1)x_1 + \tau_1(u_2)x_2 + \tau_1(u_3)x_3 + \cdots + \tau_1(u_{n+1})x_{n+1} &= 0 \\
\tau_2(u_1)x_1 + \tau_2(u_2)x_2 + \tau_2(u_3)x_3 + \cdots + \tau_2(u_{n+1})x_{n+1} &= 0 \\
\tau_3(u_1)x_1 + \tau_3(u_2)x_2 + \tau_3(u_3)x_3 + \cdots + \tau_3(u_{n+1})x_{n+1} &= 0 \\
&\vdots \\
\tau_n(u_1)x_1 + \tau_n(u_2)x_2 + \tau_n(u_3)x_3 + \cdots + \tau_n(u_{n+1})x_{n+1} &= 0.
\end{align*} \]

Such a system \((n\) homogeneous equations in \(n+1\) unknowns) has a nontrivial solution as will be shown in Exercise VII.2.4(d) (see also Lemma 5.1.1 of *Real Analysis with an Introduction to Wavelets*, Don Hong, Jianzhong Wang, and Robert Gardner, Academic Press/Elsevier Press, 2005).
Lemma V.2.9 (continued 1)

\[ \tau_1(u_1)x_1 + \tau_1(u_2)x_2 + \tau_1(u_3)x_3 + \cdots + \tau_1(u_{n+1})x_{n+1} = 0 \]
\[ \tau_2(u_1)x_1 + \tau_2(u_2)x_2 + \tau_2(u_3)x_3 + \cdots + \tau_2(u_{n+1})x_{n+1} = 0 \]
\[ \tau_3(u_1)x_1 + \tau_3(u_2)x_2 + \tau_3(u_3)x_3 + \cdots + \tau_3(u_{n+1})x_{n+1} = 0 \]
\[ \vdots \]
\[ \tau_n(u_1)x_1 + \tau_n(u_2)x_2 + \tau_n(u_3)x_3 + \cdots + \tau_n(u_{n+1})x_{n+1} = 0. \]

Such a system \((n\) homogeneous equations in \(n + 1\) unknowns) has a nontrivial solution as will be shown in Exercise VII.2.4(d) (see also Lemma 5.1.1 of *Real Analysis with an Introduction to Wavelets*, Don Hong, Jianzhong Wang, and Robert Gardner, Academic Press/Elsevier Press, 2005). Among all such nontrivial solutions choose one, say \(x_1 = a_1, x_2 = a_2, \ldots, x_{n+1} = a_{n+1}\) with a minimal number of nonzero \(a_i\).

By reindexing if necessary we may assume that \(x_1 = a_1, x_2 = a_2, \ldots, x_r = a_r\) and \(x_{r+1} = x_{r+1} = \cdots x_{n+1} = 0\) where \(a_r \neq 0\). Since each multiple of a solution is also a solution then we may also assume \(a_1 = 1_F\).
Lemma V.2.9 (continued 1)

\[ \tau_1(u_1)x_1 + \tau_1(u_2)x_2 + \tau_1(u_3)x_3 + \cdots + \tau_1(u_{n+1})x_{n+1} = 0 \]
\[ \tau_2(u_1)x_1 + \tau_2(u_2)x_2 + \tau_2(u_3)x_3 + \cdots + \tau_2(u_{n+1})x_{n+1} = 0 \]
\[ \tau_3(u_1)x_1 + \tau_3(u_2)x_2 + \tau_3(u_3)x_3 + \cdots + \tau_3(u_{n+1})x_{n+1} = 0 \]
\[ \vdots \]
\[ \tau_n(u_1)x_1 + \tau_n(u_2)x_2 + \tau_n(u_3)x_3 + \cdots + \tau_n(u_{n+1})x_{n+1} = 0. \]

Such a system \((n\) homogeneous equations in \(n + 1\) unknowns\) has a nontrivial solution as will be shown in Exercise VII.2.4(d) (see also Lemma 5.1.1 of \textit{Real Analysis with an Introduction to Wavelets}, Don Hong, Jianzhong Wang, and Robert Gardner, Academic Press/Elsevier Press, 2005). Among all such nontrivial solutions choose one, say \(x_1 = a_1, x_2 = a_2, \ldots, x_{n+1} = a_{n+1}\) with a minimal number of nonzero \(a_i\). By reindexing if necessary we may assume that \(x_1 = a_1, x_2 = a_2, \ldots, x_r = a_r\) and \(x_{r+1} = x_{r+1} = \cdots x_{n+1} = 0\) where \(a_r \neq 0\). Since each multiple of a solution is also a solution then we may also assume \(a_1 = 1_F\).
Lemma V.2.9 (continued 2)

Proof (continued). In the conclusion of the proof below, we will show that the hypothesis that $u_1, u_2, \ldots, u_{n+1} \in H'$ are linearly independent over $J'$ implies that there exists $\sigma \in J$ such that

$$x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r \text{ and } x_{r+1} = x_{r+2} = \cdots = x_{n+1} = 0$$

is also a nontrivial solution to the system of equations (1) and $\sigma a_2 = a_2$. Since the difference of two solutions is also a solution (since the system (1) is linear and homogeneous) then

$$x_1 = a_1 - \sigma a_1, x_2 = a_2 - \sigma a_2, \ldots, x_t = a_r - \sigma a_r, \text{ and } x_{r+1} = x_{r+2} = \cdots = x_{n+1} = 0$$

is also a solution of the system of equations (1).
Lemma V.2.9 (continued 2)

Proof (continued). In the conclusion of the proof below, we will show that the hypothesis that \( u_1, u_2, \ldots, u_{n+1} \in H' \) are linearly independent over \( J' \) implies that there exists \( \sigma \in J \) such that
\[
\begin{align*}
x_1 &= \sigma a_1, \\
x_2 &= \sigma a_2, \\
&\vdots \\
x_r &= \sigma a_r \\
x_{r+1} &= x_{r+2} = \cdots = x_{n+1} = 0
\end{align*}
\]
is also a nontrivial solution to the system of equations (1) and \( \sigma a_2 = a_2 \).
Since the difference of two solutions is also a solution (since the system (1) is linear and homogeneous) then
\[
\begin{align*}
x_1 &= a_1 - \sigma a_1, \\
x_2 &= a_2 - \sigma a_2, \\
&\vdots \\
x_t &= a_r - \sigma a_r, \\
x_{r+1} &= x_{r+2} = \cdots = x_{n+1} = 0
\end{align*}
\]
is also a solution of the system of equations (1). But since
\[
a_1 - \sigma a_1 = 1_F - 1_F = 0 \quad (\sigma \in J < \text{Aut}_K(F) \text{ implies that } \sigma \text{ fixes the elements of } K, \text{ including the multiplicative identity}) \quad \text{and} \quad a_2 \neq \sigma a_2
\]
then
\[
\begin{align*}
x_1 &= 0, \\
x_2 &= a_2 - \sigma a_2 \neq 0, \\
x_3 &= a_3 - \sigma a_3, \\
&\vdots \\
x_r &= a_r - \sigma a_r \\
x_{r+1} &= x_{r+2} = \cdots = x_{n+1} = 0
\end{align*}
\]
is a nontrivial solution of the system of equations (1) (since \( x_2 \neq 0 \)) with at most \( r - 1 \) nonzero entries, a CONTRADICTION to the minimality of \( r \) of nonzero terms is a nontrivial solution to the system of equations (1).
Lemma V.2.9 (continued 2)

Proof (continued). In the conclusion of the proof below, we will show that the hypothesis that \( u_1, u_2, \ldots, u_{n+1} \in H' \) are linearly independent over \( J' \) implies that there exists \( \sigma \in J \) such that

\[
\begin{align*}
x_1 &= \sigma a_1, \\ x_2 &= \sigma a_2, \\ &\vdots \\ x_r &= \sigma a_r \\
x_{r+1} = x_{r+2} = \cdots = x_{n+1} &= 0
\end{align*}
\]

is also a nontrivial solution to the system of equations (1) and \( \sigma a_2 = a_2 \).

Since the difference of two solutions is also a solution (since the system (1) is linear and homogeneous) then

\[
\begin{align*}
x_1 &= a_1 - \sigma a_1, \\ x_2 &= a_2 - \sigma a_2, \\ &\vdots \\ x_t &= a_r - \sigma a_r, \\
x_{r+1} = x_{r+2} = \cdots = x_{n+1} &= 0
\end{align*}
\]

is also a solution of the system of equations (1). But since

\[
a_1 - \sigma a_1 = 1_F - 1_F = 0 \quad (\sigma \in J < \text{Aut}_K(F) \text{ implies that } \sigma \text{ fixes the elements of } K, \text{ including the multiplicative identity}) \text{ and } a_2 \neq \sigma a_2 \text{ then}
\]

\[
x_1 = 0, \quad x_2 = a_2 - \sigma a_2 \neq 0, \quad x_3 = a_3 - \sigma a_3, \quad \ldots, \quad x_r = a_r - \sigma a_r \quad \text{and}
\]

\[
x_{r+1} = x_{r+2} = \cdots = x_{n+1} = 0
\]

is a nontrivial solution of the system of equations (1) (since \( x_2 \neq 0 \)) with at most \( r - 1 \) nonzero entries, a CONTRADICTION to the minimality of \( r \) of nonzero terms is a nontrivial solution to the system of equations (1).
**Proof (continued).** This contradiction shows that the assumption $[H' : J'] > n$ is false, and hence $[H' : J'] \leq n$.

To complete the proof, we must find $\sigma \in J$ with the desired properties. Now $\{\tau_1, \tau_2, \ldots, \tau_n\}$ is a set of representatives of the cosets of $H$, then exactly one of the $\tau_j$, say $\tau_1$, is in $H$ itself.
Lemma V.2.9 (continued 3)

**Proof (continued).** This contradiction shows that the assumption \([H' : J'] > n\) is false, and hence \([H' : J'] \leq n\).

To complete the proof, we must find \(\sigma \in J\) with the desired properties. Now \(\{\tau_1, \tau_2, \ldots, \tau_n\}\) is a set of representatives of the cosets of \(H\), then exactly one of the \(\tau_j\), say \(\tau_1\), is in \(H\) itself. Since \(H' = \text{Aut}_H(F)\), then \(\tau_1\) fixes the elements of \(H'\) and so \(\tau_1(u_i) = u_i \in H'\) for all \(i = 1, 2, \ldots, n + 1\). So the first equation in the system of equations (1) becomes

\[ u_1a_1 + a_2a_2 + \cdots + u_ra_r = 0. \]

Now each \(a_i\) is nonzero for \(1 \leq i \leq r\) and the \(u_i\) are linearly independent over \(J'\).
Lemma V.2.9 (continued 3)

**Proof (continued).** This contradiction shows that the assumption 

\[ [H' : J'] > n \]

is false, and hence \([H' : J'] \leq n\).

To complete the proof, we must find \(\sigma \in J\) with the desired properties. Now \(\{\tau_1, \tau_2, \ldots, \tau_n\}\) is a set of representatives of the cosets of \(H\), then exactly one of the \(\tau_j\), say \(\tau_1\), is in \(H\) itself. Since \(H' = \text{Aut}_H(F)\), then \(\tau_1\) fixes the elements of \(H'\) and so \(\tau_1(u_i) = u_i \in H'\) for all \(i = 1, 2, \ldots, n + 1\).

So the first equation in the system of equations (1) becomes

\[ u_1 a_1 + a_2 a_2 + \cdots + u_r a_r = 0. \]

Now each \(a_i\) is nonzero for \(1 \leq i \leq r\) and the \(u_i\) are linearly independent over \(J'\). So it must be that some \(a_i\) is not in \(J'\), say \(a_2 \not\in J'\). Since \(J'\) is the fixed field of \(J\), then there is some \(\sigma \in J\) such that \(\sigma a_2 \neq a_2\) (that is, \(\sigma\) does not fix \(a_2\)).
Lemma V.2.9 (continued 3)

Proof (continued). This contradiction shows that the assumption 
$[H' : J'] > n$ is false, and hence $[H' : J'] \leq n$.

To complete the proof, we must find $\sigma \in J$ with the desired properties. 
Now $\{\tau_1, \tau_2, \ldots, \tau_n\}$ is a set of representatives of the cosets of $H$, then 
extactly one of the $\tau_j$, say $\tau_1$, is in $H$ itself. Since $H' = \text{Aut}_H(F)$, then $\tau_1$ 
fixes the elements of $H'$ and so $\tau_1(u_i) = u_i \in H'$ for all $i = 1, 2, \ldots, n + 1$. 
So the first equation in the system of equations (1) becomes 
$u_1a_1 + a_2a_2 + \cdots + u_ra_r = 0$. Now each $a_i$ is nonzero for $1 \leq i \leq r$ and 
the $u_i$ are linearly independent over $J'$. So it must be that some $a_i$ is not 
in $J'$, say $a_2 \notin J'$. Since $J'$ is the fixed field of $J$, then there is some $\sigma \in J$ 
such that $\sigma a_2 \neq a_2$ (that is, $\sigma$ does not fix $a_2$).
Lemma V.2.9 (continued 4)

Proof (continued). Next, consider a second system of equations (which we will show to be equivalent to [that is, have the identical solutions as] the first system of equations (1)):

\[
\begin{align*}
\sigma \tau_1(u_1)x_1 + \sigma \tau_1(u_2)x_2 + \sigma \tau_1(u_3)x_3 + \cdots + \sigma \tau_1(u_{n+1})x_{n+1} &= 0 \\
\sigma \tau_2(u_1)x_1 + \sigma \tau_2(u_2)x_2 + \sigma \tau_2(u_3)x_3 + \cdots + \sigma \tau_2(u_{n+1})x_{n+1} &= 0 \\
\sigma \tau_3(u_1)x_1 + \sigma \tau_3(u_2)x_2 + \sigma \tau_3(u_3)x_3 + \cdots + \sigma \tau_3(u_{n+1})x_{n+1} &= 0 \\
&\vdots \\
\sigma \tau_n(u_1)x_1 + \sigma \tau_n(u_2)x_2 + \sigma \tau_n(u_3)x_3 + \cdots + \sigma \tau_n(u_{n+1})x_{n+1} &= 0.
\end{align*}
\]

Since \( \sigma \in J < \text{Aut}_K(F) \) then \( \sigma(0) = 0 \) and if we apply \( \sigma \) to each of the equations in the first system (1), then we get the second system (2). Since \( x_1 = 1_1, x_2 = a_2, \ldots, x_r = a_r \) and \( x_{r+1} = x_{r+2} = \cdots x_{n+1} = 0 \) is a solution of system (1), then \( x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r \) and \( x_{r+1} = x_{r+2} = \cdots = x_{n+1} = 0 \) is a solution of system (2).
Lemma V.2.9 (continued 4)

Proof (continued). Next, consider a second system of equations (which we will show to be equivalent to [that is, have the identical solutions as] the first system of equations (1)):

\[
\begin{align*}
    \sigma \tau_1(u_1)x_1 + \sigma \tau_1(u_2)x_2 + \sigma \tau_1(u_3)x_3 + \cdots + \sigma \tau_1(u_{n+1})x_{n+1} &= 0 \\
    \sigma \tau_2(u_1)x_1 + \sigma \tau_2(u_2)x_2 + \sigma \tau_2(u_3)x_3 + \cdots + \sigma \tau_2(u_{n+1})x_{n+1} &= 0 \\
    \sigma \tau_3(u_1)x_1 + \sigma \tau_3(u_2)x_2 + \sigma \tau_3(u_3)x_3 + \cdots + \sigma \tau_3(u_{n+1})x_{n+1} &= 0 \\
    & \vdots \\
    \sigma \tau_n(u_1)x_1 + \sigma \tau_n(u_2)x_2 + \sigma \tau_n(u_3)x_3 + \cdots + \sigma \tau_n(u_{n+1})x_{n+1} &= 0.
\end{align*}
\]

Since \( \sigma \notin J < \text{Aut}_K(F) \) then \( \sigma(0) = 0 \) and if we apply \( \sigma \) to each of the equations in the first system (1), then we get the second system (2). Since \( x_1 = 1_1, x_2 = a_2, \ldots, x_r = a_r \) and \( x_{r+1} = x_{r+2} = \cdots = x_{n+1} = 0 \) is a solution of system (1), then \( x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r \) and \( x_{r+1} = x_{r+2} = \cdots = x_{n+1} = 0 \) is a solution of system (2).
Lemma V.2.9 (continued 5)

Proof (continued). We claim that system (2), except for the order of the equations, is identical with system (1) (so that 
\[ x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r \text{ and } x_{r+1} = x_{r+1} = \cdots = x_{n+1} = 0 \]
is a solution of system (1); this will show that \( \sigma \) satisfies the conditions mentioned above). We make two claims:

(i) For any \( \sigma \in J \), the set \( \{ \sigma \tau_1, \sigma \tau_2, \ldots, \sigma \tau_n \} \subset J \) is a complete set of coset representatives of the cosets of \( H \) in \( J \).
Lemma V.2.9 (continued 5)

Proof (continued). We claim that system (2), except for the order of the equations, is identical with system (1) (so that 
\[ x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r \text{ and } x_{r+1} = x_{r+1} = \cdots = x_{n+1} = 0 \]
is a solution of system (1); this will show that \( \sigma \) satisfies the conditions mentioned above). We make two claims:

(i) For any \( \sigma \in J \), the set \( \{\sigma \tau_1, \sigma \tau_2, \ldots, \sigma \tau_n\} \subset J \) is a complete set of coset representatives of the cosets of \( H \) in \( J \).

Sub-Proof. First, since each \( \tau_i \in J \) and \( \sigma \in J \), then \( \sigma \tau_i \in J \). Now 
\[ \sigma \tau_i H = \sigma \tau_i H \] if and only if \((\sigma \tau_i)^{-1}(\sigma \tau_j) \in H\) by Theorem I.4.3(iii); that is, 
\[ \tau_i^{-1} \sigma^{-1} \sigma \tau_j = \tau_i^{-1} \tau_j \in H. \]
Proof (continued). We claim that system (2), except for the order of the equations, is identical with system (1) (so that

\[ x_1 = \sigma a_1, \ x_2 = \sigma a_2, \ldots, \ x_r = \sigma a_r \text{ and } x_{r+1} = x_{r+1} = \cdots = x_{n+1} = 0 \]

is a solution of system (1); this will show that \( \sigma \) satisfies the conditions mentioned above). We make two claims:

(i) For any \( \sigma \in J \), the set \( \{\sigma \tau_1, \sigma \tau_2, \ldots, \sigma \tau_n\} \subset J \) is a complete set of coset representatives of the cosets of \( H \) in \( J \).

Sub-Proof. First, since each \( \tau_i \in J \) and \( \sigma \in J \), then \( \sigma \tau_i \in J \). Now

\[ \sigma \tau_i H = \sigma \tau_i H \text{ if and only if } (\sigma \tau_i)^{-1}(\sigma \tau_j) \in H \] by Theorem I.4.3(iii); that is,

\[ \tau_i^{-1} \sigma^{-1} \sigma \tau_j = \tau_i^{-1} \tau_j \in H. \]

Again by Theorem I.4.3(iii), \( \tau_i^{-1} \tau_j \in H \) if only if \( \tau_i H = \tau_j H \). So \( \sigma \tau_i H = \sigma \tau_j H \) if and only if \( \tau_i H = \tau_j H \). Since

\( \{\tau_1, \tau_2, \ldots, \tau_n\} \) is a complete set of representatives of the left cosets of \( H \) in \( J \), then so is \( \{\sigma \tau_1, \sigma \tau_2, \ldots, \sigma \tau_n\} \). Sub-Q.E.D.
Proof (continued). We claim that system (2), except for the order of the equations, is identical with system (1) (so that 
\[ x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r \text{ and } x_{r+1} = x_{r+1} = \cdots = x_{n+1} = 0 \]
is a solution of system (1); this will show that \( \sigma \) satisfies the conditions mentioned above). We make two claims:

(i) For any \( \sigma \in J \), the set \( \{\sigma \tau_1, \sigma \tau_2, \ldots, \sigma \tau_n\} \subset J \) is a complete set of coset representatives of the cosets of \( H \) in \( J \).

Sub-Proof. First, since each \( \tau_i \in J \) and \( \sigma \in J \), then \( \sigma \tau_i \in J \). Now 
\[ \sigma \tau_i H = \sigma \tau_i H \text{ if and only if } (\sigma \tau_i)^{-1} (\sigma \tau_j) \in H \text{ by Theorem I.4.3(iii); that is, } \tau_i^{-1} \sigma^{-1} \sigma \tau_j = \tau_i^{-1} \tau_j \in H. \]
Again by Theorem I.4.3(iii), \( \tau_i^{-1} \tau_j \in H \) if only if \( \tau_i H = \tau_j H \). So \( \sigma \tau_i H = \sigma \tau_j H \) if and only if \( \tau_i H = \tau_j H \). Since 
\( \{\tau_1, \tau_2, \ldots, \tau_n\} \) is a complete set of representatives of the left cosets of \( H \) in \( J \), then so is \( \{\sigma \tau_1, \sigma \tau_2, \ldots, \sigma \tau_n\} \). Sub-Q.E.D.
Lemma V.2.9 (continued 6)

Proof (continued).

(ii) If $\zeta$ and $\theta$ are both elements in the same coset of $H$ in $J$, then (since $u_i \in H'$) $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots n + 1$.

Sub-Proof. Let $\zeta, \theta \in aH$. Then $\zeta = ah_1$ and $\theta = ah_2$ for some $h_1, h_2 \in H$. 

Lemma V.2.9 (continued 6)

Proof (continued).

(ii) If \( \zeta \) and \( \theta \) are both elements in the same coset of \( H \) in \( J \), then (since \( u_i \in H' \)) \( \zeta(u_i) = \theta(u_i) \) for \( i = 1, 2, \ldots n + 1 \).

Sub-Proof. Let \( \zeta, \theta \in aH \). Then \( \zeta = ah_1 \) and \( \theta = ah_2 \) for some \( h_1, h_2 \in H \). Since \( H' \) is the fixed field of \( H \) and each \( u_i \in H' \), then
\[
\zeta(u_i) = (ah_1)(u_i) = ah_1(u_i) = au_i \quad \text{and} \quad \theta(u_i) = (ah_2)(u_i) = ah_2(u_i) = au_i.
\]
So \( \zeta(u_i) = \theta(u_i) \) for \( i = 1, 2, \ldots, n + 1 \). Sub-Q.E.D.
Lemma V.2.9 (continued 6)

Proof (continued).

(ii) If $\zeta$ and $\theta$ are both elements in the same coset of $H$ in $J$, then (since $u_i \in H'$) $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots, n + 1$.

Sub-Proof. Let $\zeta, \theta \in aH$. Then $\zeta = ah_1$ and $\theta = ah_2$ for some $h_1, h_2 \in H$. Since $H'$ is the fixed field of $H$ and each $u_i \in H'$, then $\zeta(u_i) = (ah_1)(u_i) = ah_1(u_i) = au_i$ and $\theta(u_i) = (ah_2)(u_i) = ah_2(u_i) = au_i$. So $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots, n + 1$. Sub-Q.E.D.

It now follows from claim (i) that there is some reordering $i_1, i_2, \ldots, i_{n+1}$ of $1, 2, \ldots, n + 1$ so that for each $k = 1, 2, \ldots, n + 1$, $\sigma \tau_k$ and $\tau_{i_k}$ are in the same coset of $H$ in $J$. By (ii), the $k$th equation of system (2) (with coefficients $\sigma \tau_k(u_i)$) is identical with the $i_k$th equation of system (1) (with coefficients $\tau_{i_k}(u_i)$).
Lemma V.2.9 (continued 6)

Proof (continued).

(ii) If $\zeta$ and $\theta$ are both elements in the same coset of $H$ in $J$, then (since $u_i \in H'$) $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots, n + 1$.

Sub-Proof. Let $\zeta, \theta \in aH$. Then $\zeta = ah_1$ and $\theta = ah_2$ for some $h_1, h_2 \in H$. Since $H'$ is the fixed field of $H$ and each $u_i \in H'$, then $\zeta(u_i) = (ah_1)(u_i) = ah_1(u_i) = au_i$ and $\theta(u_i) = (ah_2)(u_i) = ah_2(u_i) = au_i$. So $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots, n + 1$. Sub-Q.E.D.

It now follows from claim (i) that there is some reordering $i_1, i_2, \ldots, i_{n+1}$ of $1, 2, \ldots, n + 1$ so that for each $k = 1, 2, \ldots, n + 1$, $\sigma \tau_k$ and $\tau_{i_k}$ are in the same coset of $H$ in $J$. By (ii), the $k$th equation of system (2) (with coefficients $\sigma \tau_k(u_i)$) is identical with the $i_k$th equation of system (1) (with coefficients $\tau_{i_k}(u_i)$). So we have, in particular, that the solution $x_1 = a_1, x_2 = a_2, \ldots, x_r = a_r$ and $x_{r+1} = x_{r+2} = \cdots x_{n+1} = 0$ of system (2) is also a solution of system (1).
Lemma V.2.9 (continued 6)

Proof (continued).

(ii) If $\zeta$ and $\theta$ are both elements in the same coset of $H$ in $J$, then (since $u_i \in H'$) $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots, n + 1$.

Sub-Proof. Let $\zeta, \theta \in aH$. Then $\zeta = ah_1$ and $\theta = ah_2$ for some $h_1, h_2 \in H$. Since $H'$ is the fixed field of $H$ and each $u_i \in H'$, then $\zeta(u_i) = (ah_1)(u_i) = ah_1(u_i) = au_i$ and $\theta(u_i) = (ah_2)(u_i) = ah_2(u_i) = au_i$. So $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots, n + 1$. Sub-Q.E.D.

It now follows from claim (i) that there is some reordering $i_1, i_2, \ldots, i_{n+1}$ of $1, 2, \ldots, n + 1$ so that for each $k = 1, 2, \ldots, n + 1$, $\sigma \tau_k$ and $\tau_{i_k}$ are in the same coset of $H$ in $J$. By (ii), the $k$th equation of system (2) (with coefficients $\sigma \tau_k(u_i)$) is identical with the $i_k$th equation of system (1) (with coefficients $\tau_{i_k}(u_i)$). So we have, in particular, that the solution $x_1 = a_1, x_2 = a_2, \ldots, x_r = a_r$ and $x_{r+1} = x_{r+2} = \cdots x_{n+1} = 0$ of system (2) is also a solution of system (1). We can now pick up at step $(\ast)$ and complete the proof by contradiction. \(\square\)
Lemma V.2.9 (continued 6)

Proof (continued).

(ii) If $\zeta$ and $\theta$ are both elements in the same coset of $H$ in $J$, then (since $u_i \in H'$) $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots, n + 1$.

Sub-Proof. Let $\zeta, \theta \in aH$. Then $\zeta = ah_1$ and $\theta = ah_2$ for some $h_1, h_2 \in H$. Since $H'$ is the fixed field of $H$ and each $u_i \in H'$, then $\zeta(u_i) = (ah_1)(u_i) = ah_1(u_i) = au_i$ and $\theta(u_i) = (ah_2)(u_i) = ah_2(u_i) = au_i$. So $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots, n + 1$. Sub-Q.E.D.

It now follows from claim (i) that there is some reordering $i_1, i_2, \ldots, i_{n+1}$ of $1, 2, \ldots, n + 1$ so that for each $k = 1, 2, \ldots, n + 1$, $\sigma \tau_k$ and $\tau_{i_k}$ are in the same coset of $H$ in $J$. By (ii), the $k$th equation of system (2) (with coefficients $\sigma \tau_k(u_i)$) is identical with the $i_k$th equation of system (1) (with coefficients $\tau_{i_k}(u_i)$). So we have, in particular, that the solution $x_1 = a_1, x_2 = a_2, \ldots, x_r = a_r$ and $x_{r+1} = x_{r+2} = \cdots x_{n+1} = 0$ of system (2) is also a solution of system (1). We can now pick up at step (\*) and complete the proof by contradiction. \qed
Lemma V.2.10

Lemma V.2.10. Let $F$ be an extension field of $K$, $L$ and $M$ intermediate fields with $L \subset M$, and $H$, $J$ subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$.

(i) If $L$ is closed and $[M : L]$ finite, then $M$ is closed and $[L' : M'] = [M : L]$;

(ii) if $H$ is closed and $[J : H]$ finite, then $J$ is closed and $[H' : J'] = [J : H]$;

(iii) if $F$ is a finite dimensional Galois extension of $K$, then all intermediate fields and and all subgroups of the Galois group are closed and $\text{Aut}_K(F)$ has order $[F : K]$.

Proof. (i) By Lemma V.2.6(iii), $M \subset M''$. Since $L \subset M \subset M''$, by Theorem V.1.2 we have $[M'' : L] = [M'' : M][M : L]$ and so $[M : L] \leq [M'' : L]$. Now $[L' : M'] \leq [M : L]$ By Lemma V.2.8 and $[M'' : L''] \leq [L' : M']$ by Lemma V.2.9.
Lemma V.2.10

**Lemma V.2.10.** Let $F$ be an extension field of $K$, $L$ and $M$ intermediate fields with $L \subset M$, and $H, J$ subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$.

(i) If $L$ is closed and $[M : L]$ finite, then $M$ is closed and $[L' : M'] = [M : L]$;

(ii) if $H$ is closed and $[J : H]$ finite, then $J$ is closed and $[H' : J'] = [J : H]$;

(iii) if $F$ is a finite dimensional Galois extension of $K$, then all intermediate fields and and all subgroups of the Galois group are closed and $\text{Aut}_K(F)$ has order $[F : K]$.

**Proof. (i)** By Lemma V.2.6(iii), $M \subset M''$. Since $L \subset M \subset M''$, by Theorem V.1.2 we have $[M'' : L] = [M'' : M][M : L]$ and so $[M : L] \leq [M'' : L]$. Now $[L' : M'] \leq [M : L]$ By Lemma V.2.8 and $[M'' : L''] \leq [L' : M']$ by Lemma V.2.9.
Lemma V.2.10 (continued 1)

Lemma V.2.10. Let $F$ be an extension field of $K$, $L$ and $M$ intermediate fields with $L \subset M$, and $H, J$ subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$.

(i) If $L$ is closed and $[M : L]$ finite, then $M$ is closed and $[L' : M'] = [M : L]$.

Proof (continued). (i) Combining these inequalities gives

\[
[M : L] \leq [M'' : L] = [M'' : L''] \text{ since } L'' = L \\
\leq [L' : M'] \leq [M : L].
\]

Therefore the inequalities reduce to equalities and $[L' : M'] = [M : L]$. Also, $[M'' : L] = [M : L]$ so the dimension of $M''$ over $L$ is the same as the dimension of $M$ over $L$. Also, by Lemma V.2.6(iii), $M \subset M''$ and so $M = M''$ and $M$ is closed.
Lemma V.2.10. Let \( F \) be an extension field of \( K, L \) and \( M \) intermediate fields with \( L \subset M \), and \( H, J \) subgroups of the Galois group \( \text{Aut}_K(F) \) with \( H < J \).

(i) If \( L \) is closed and \( [M : L] \) finite, then \( M \) is closed and \( [L' : M'] = [M : L] \).

Proof (continued). (i) Combining these inequalities gives

\[
[M : L] \leq [M'' : L] = [M'' : L''] \quad \text{since} \quad L'' = L \\
\leq [L' : M'] \leq [M : L].
\]

Therefore the inequalities reduce to equalities and \( [L' : M'] = [M : L] \).

Also, \( [M'' : L] = [M : L] \) so the dimension of \( M'' \) over \( L \) is the same as the dimension of \( M \) over \( L \). Also, by Lemma V.2.6(iii), \( M \subset M'' \) and so \( M = M'' \) and \( M \) is closed.
Lemma V.2.10. Let $F$ be an extension field of $K$, $L$ and $M$ intermediate fields with $L \subset M$, and $H, J$ subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$.

(ii) if $H$ is closed and $[J : H]$ finite, then $J$ is closed and $[H' : J'] = [J : H]$.

Proof. (ii) By Lemma V.2.6(iii), $J < J''$. Since $H < J < J''$ then the number of cosets of $H$ in $J$, $[J : H]$ is less than or equal to the number of cosets of $H$ in $J''$, $[J'' : H]$; that is, $[J : H] \leq [J'' : H]$. 
Lemma V.2.10 (continued 2)

Lemma V.2.10. Let $F$ be an extension field of $K$, $L$ and $M$ intermediate fields with $L \subset M$, and $H, J$ subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$.

(ii) if $H$ is closed and $[J : H]$ finite, then $J$ is closed and $[H' : J'] = [J : H]$.

Proof. (ii) By Lemma V.2.6(iii), $J < J''$. Since $H < J < J''$ then the number of cosets of $H$ in $J$, $[J : H]$ is less than or equal to the number of cosets of $H$ in $J''$, $[J'' : H]$; that is, $[J : H] \leq [J'' : H]$. So

\[ [J : H] \leq [J'' : H] = [J'' : H''] \text{ since } H = H'' \]
\[ \leq [H' : J'] \text{ by Lemma V.2.8} \]
\[ \leq [J : H] \text{ by Lemma V.2.9}. \]

So we have $[H' : J'] = [J : H]$ as claimed.
Lemma V.2.10 (continued 2)

**Lemma V.2.10.** Let $F$ be an extension field of $K$, $L$ and $M$ intermediate fields with $L \subset M$, and $H$, $J$ subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$.

(iii) if $H$ is closed and $[J : H]$ finite, then $J$ is closed and $[H' : J'] = [J : H]$.

**Proof. (ii)** By Lemma V.2.6(iii), $J < J''$. Since $H < J < J''$ then the number of cosets of $H$ in $J$, $[J : H]$ is less than or equal to the number of cosets of $H$ in $J''$, $[J'' : H]$; that is, $[J : H] \leq [J'' : H]$. So

$$[J : H] \leq [J'' : H] = [J'' : H''] \text{ since } H = H''$$

$$\leq [H' : J'] \text{ by Lemma V.2.8}$$

$$\leq [J : H] \text{ by Lemma V.2.9.}$$

So we have $[H' : J'] = [J : H]$ as claimed. Also, $[J'' : H] = [J : H]$ and so the number of cosets of $H$ in $J$ equals the number of cosets of $H$ in $J''$. Therefore $|J| = |J''|$; also $J \subset J''$ so we must have $J = J''$. 
Lemma V.2.10. Let $F$ be an extension field of $K$, $L$ and $M$ intermediate fields with $L \subset M$, and $H, J$ subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$.

(ii) if $H$ is closed and $[J : H]$ finite, then $J$ is closed and 

Proof. (ii) By Lemma V.2.6(iii), $J < J''$. Since $H < J < J''$ then the number of cosets of $H$ in $J$, $[J : H]$ is less than or equal to the number of cosets of $H$ in $J''$, $[J'' : H]$; that is, $[J : H] \leq [J'' : H]$. So

$$[J : H] \leq [J'' : H] = [J'' : H''] \text{ since } H = H''$$

$$\leq [H' : J'] \text{ by Lemma V.2.8}$$

$$\leq [J : H] \text{ by Lemma V.2.9.}$$

So we have $[H' : J'] = [J : H]$ as claimed. Also, $[J'' : H] = [J : H]$ and so the number of cosets of $H$ in $J$ equals the number of cosets of $H$ in $J''$. Therefore $|J| = |J''|$; also $J \subset J''$ so we must have $J = J''$. 
Lemma V.2.10 (continued 3)

**Lemma V.2.10.** Let $F$ be an extension field of $K$, $L$ and $M$ intermediate fields with $L \subset M$, and $H, J$ subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$. 

(iii) if $F$ is a finite dimensional Galois extension of $K$, then all intermediate fields and all subgroups of the Galois group are closed and $\text{Aut}_K(F)$ has order $[F : K]$.

**Proof.** (iii) If $E$ is an intermediate field, $K \subset E \subset F$, then $[F : K] = [F : E][E : K]$ by Theorem V.1.2 and since $[F : K]$ is hypothesized to be finite, then $[E : K]$ is finite.
Lemma V.2.10. Let $F$ be an extension field of $K$, $L$ and $M$ intermediate fields with $L \subset M$, and $H, J$ subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$. (iii) if $F$ is a finite dimensional Galois extension of $K$, then all intermediate fields and and all subgroups of the Galois group are closed and $\text{Aut}_K(F)$ has order $[F : K]$.

Proof. (iii) If $E$ is an intermediate field, $K \subset E \subset F$, then $[F : K] = [F : E][E : K]$ by Theorem V.1.2 and since $[F : K]$ is hypothesized to be finite, then $[E : K]$ is finite. Since $F$ is Galois over $K$ then $K$ is closed (see the note on page 246 right after the definition of closed). So every intermediate field is closed.
Lemma V.2.10. Let $F$ be an extension field of $K$, $L$ and $M$ intermediate fields with $L \subset M$, and $H, J$ subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$. (iii) if $F$ is a finite dimensional Galois extension of $K$, then all intermediate fields and and all subgroups of the Galois group are closed and $\text{Aut}_K(F)$ has order $[F : K]$.

Proof. (iii) If $E$ is an intermediate field, $K \subset E \subset F$, then 
$[F : K] = [F : E][E : K]$ by Theorem V.1.2 and since $[F : K]$ is hypothesized to be finite, then $[E : K]$ is finite. Since $F$ is Galois over $K$ then $K$ is closed (see the note on page 246 right after the definition of closed). So every intermediate field is closed. Now (i) (with $L = K$ and $M = E$) implies that $E$ is closed and $[K' : E'] = [E : K]$. In particular, if $E = F$ then $|\text{Aut}_K(F)| = [\text{Aut}_K(E) : 1] = [K' : F'] = [F : K]$ is finite. Therefore, every subgroup $J$ of $\text{Aut}_K(F)$ is finite.
Lemma V.2.10. Let $F$ be an extension field of $K$, $L$ and $M$ intermediate fields with $L \subset M$, and $H, J$ subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$. (iii) if $F$ is a finite dimensional Galois extension of $K$, then all intermediate fields and and all subgroups of the Galois group are closed and $\text{Aut}_K(F)$ has order $[F : K]$.

Proof. (iii) If $E$ is an intermediate field, $K \subset E \subset F$, then $[F : K] = [F : E][E : K]$ by Theorem V.1.2 and since $[F : K]$ is hypothesized to be finite, then $[E : K]$ is finite. Since $F$ is Galois over $K$ then $K$ is closed (see the note on page 246 right after the definition of closed). So every intermediate field is closed. Now (i) (with $L = K$ and $M = E$) implies that $E$ is closed and $[K' : E'] = [E : K]$. In particular, if $E = F$ then $|\text{Aut}_K(F)| = [\text{Aut}_K(E) : 1] = [K' : F'] = [F : K]$ is finite. Therefore, every subgroup $J$ of $\text{Aut}_K(F)$ is finite. Now $1' = F$ and $1'' = F' = \text{Aut}_F(F) = 1$, so 1 is closed. Now by (ii), $J$ is closed and so every subgroup of $\text{Aut}_K(F)$ is closed. 

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Lemma V.2.10 (continued 3)

**Lemma V.2.10.** Let $F$ be an extension field of $K$, $L$ and $M$ intermediate fields with $L \subset M$, and $H, J$ subgroups of the Galois group $\text{Aut}_K(F)$ with $H < J$. (iii) if $F$ is a finite dimensional Galois extension of $K$, then all intermediate fields and and all subgroups of the Galois group are closed and $\text{Aut}_K(F)$ has order $[F : K]$.

**Proof.** (iii) If $E$ is an intermediate field, $K \subset E \subset F$, then $[F : K] = [F : E][E : K]$ by Theorem V.1.2 and since $[F : K]$ is hypothesized to be finite, then $[E : K]$ is finite. Since $F$ is Galois over $K$ then $K$ is closed (see the note on page 246 right after the definition of closed). So every intermediate field is closed. Now (i) (with $L = K$ and $M = E$) implies that $E$ is closed and $[K' : E'] = [E : K]$. In particular, if $E = F$ then $|\text{Aut}_K(F)| = [\text{Aut}_K(E) : 1] = [K' : F'] = [F : K]$ is finite. Therefore, every subgroup $J$ of $\text{Aut}_K(F)$ is finite. Now $1' = F$ and $1'' = F' = \text{Aut}_F(F) = 1$, so $1$ is closed. Now by (ii), $J$ is closed and so every subgroup of $\text{Aut}_K(F)$ is closed. 

Lemma V.2.11. Let $F$ be an extension field of $K$.

(i) If $E$ is a stable intermediate field of the extension, then $E' = \text{Aut}_E(F)$ is a normal subgroup of the Galois group $\text{Aut}_K(F)$;

(ii) if $H$ is a normal subgroup of $\text{Aut}_K(F)$, then the fixed field $H'$ of $H$ is a stable intermediate field of the extension.

Proof. (i) If $u \in E$ and $\sigma \in \text{Aut}_K(F)$ then $\sigma(u) \in E$ by the stability of $E$. Hence for $\tau \in E' = \text{Aut}_E(F)$ we have $\tau \sigma(u) = \sigma(u)$. 


Lemma V.2.11. Let $F$ be an extension field of $K$.

(i) If $E$ is a stable intermediate field of the extension, then $E' = \text{Aut}_E(F)$ is a normal subgroup of the Galois group $\text{Aut}_K(F)$;

(ii) if $H$ is a normal subgroup of $\text{Aut}_K(F)$, then the fixed field $H'$ of $H$ is a stable intermediate field of the extension.

Proof. (i) If $u \in E$ and $\sigma \in \text{Aut}_K(F)$ then $\sigma(u) \in E$ by the stability of $E$. Hence for $\tau \in E' = \text{Aut}_E(F)$ we have $\tau \sigma(u) = \sigma(u)$. Therefore, for any $\sigma \in \text{Aut}_K(F)$, $\tau \in E' = \text{Aut}_E(F)$, and $u \in E$ we have $\sigma^{-1} \tau \sigma(u) = \sigma^{-1} \sigma(u) = u$. Consequently $\sigma^{-1} \tau \sigma \in E' = \text{Aut}_E(F)$ and hence $E'$ is a normal subgroup of $\text{Aut}_K(F)$ by Theorem I.5.1(iv).
Lemma V.2.11. Let $F$ be an extension field of $K$.

(i) If $E$ is a stable intermediate field of the extension, then $E' = \text{Aut}_E(F)$ is a normal subgroup of the Galois group $\text{Aut}_K(F)$;

(ii) if $H$ is a normal subgroup of $\text{Aut}_K(F)$, then the fixed field $H'$ of $H$ is a stable intermediate field of the extension.

Proof. (i) If $u \in E$ and $\sigma \in \text{Aut}_K(F)$ then $\sigma(u) \in E$ by the stability of $E$. Hence for $\tau \in E' = \text{Aut}_E(F)$ we have $\tau \sigma(u) = \sigma(u)$. Therefore, for any $\sigma \in \text{Aut}_K(F)$, $\tau \in E' = \text{Aut}_E(F)$, and $u \in E$ we have $\sigma^{-1} \tau \sigma(u) = \sigma^{-1} \sigma(u) = u$. Consequently $\sigma^{-1} \tau \sigma \in E' = \text{Aut}_E(F)$ and hence $E'$ is a normal subgroup of $\text{Aut}_K(F)$ by Theorem I.5.1(iv).
Lemma V.2.11. Let $F$ be an extension field of $K$.

(ii) if $H$ is a normal subgroup of $\text{Aut}_K(F)$, then the fixed field $H'$ of $H$ is a stable intermediate field of the extension.

Proof. (ii) If $\sigma \in \text{Aut}_K(F)$ and $\tau \in H$, then $\sigma^{-1}\tau\sigma \in H$ since $H$ is hypothesized to be a normal subgroup of $\text{Aut}_K(F)$ (by Theorem I.5.1(iv)). Therefore, for any $u \in H'$, $\sigma^{-1}\tau\sigma(u) = u$ (since $H'$ denotes the fixed field of $H$), which implies that $\tau\sigma(u) = \sigma(u)$ for all $\tau \in H$. 
Lemma V.2.11. Let $F$ be an extension field of $K$.

(ii) if $H$ is a normal subgroup of $\text{Aut}_K(F)$, then the fixed field $H'$ of $H$ is a stable intermediate field of the extension.

**Proof.** (ii) If $\sigma \in \text{Aut}_K(F)$ and $\tau \in H$, then $\sigma^{-1}\tau\sigma \in H$ since $H$ is hypothesized to be a normal subgroup of $\text{Aut}_K(F)$ (by Theorem I.5.1(iv)). Therefore, for any $u \in H'$, $\sigma^{-1}\tau\sigma(u) = u$ (since $H'$ denotes the fixed field of $H$), which implies that $\tau\sigma(u) = \sigma(u)$ for all $\tau \in H$. This $\sigma(u) \in H'$ for any $u \in H'$ and for any $\sigma \in \text{Aut}_K(F)$. This means that $H'$ is stable relative to $K$ and $F$.\qed
Lemma V.2.11. Let $F$ be an extension field of $K$.

(ii) if $H$ is a normal subgroup of $\text{Aut}_K(F)$, then the fixed field $H'$ of $H$ is a stable intermediate field of the extension.

Proof. (ii) If $\sigma \in \text{Aut}_K(F)$ and $\tau \in H$, then $\sigma^{-1}\tau\sigma \in H$ since $H$ is hypothesized to be a normal subgroup of $\text{Aut}_K(F)$ (by Theorem I.5.1(iv)). Therefore, for any $u \in H'$, $\sigma^{-1}\tau\sigma(u) = u$ (since $H'$ denotes the fixed field of $H$), which implies that $\tau\sigma(u) = \sigma(u)$ for all $\tau \in H$. This means that $\sigma(u) \in H'$ for any $u \in H'$ and for any $\sigma \in \text{Aut}_K(F)$. This means that $H'$ is stable relative to $K$ and $F$. \qed
Lemma V.2.12. If $F$ is a Galois extension field of $K$ and $E$ is a stable intermediate field of the extension, then $E$ is Galois over $K$.

Proof. If $u \in E \setminus K$ then there exists $\sigma \in \text{Aut}_K(F)$ such that $\sigma(u) \neq u$ since $F$ is Galois over $K$ (meaning $K = (\text{Aut}_K(F))' = K'$, so $u \notin K$ implies that $u$ is not fixed by some $\sigma \in \text{Aut}_K(F)$).
Lemma V.2.12. If $F$ is a Galois extension field of $K$ and $E$ is a stable intermediate field of the extension, then $E$ is Galois over $K$.

Proof. If $u \in E \setminus K$ then there exists $\sigma \in \text{Aut}_K(F)$ such that $\sigma(u) \neq u$ since $F$ is Galois over $K$ (meaning $K = (\text{Aut}_K(F))' = K'$, so $u \notin K$ implies that $u$ is not fixed by some $\sigma \in \text{Aut}_K(F)$). Since $E$ is stable then $\sigma$ maps $E$ into itself; that is, $\sigma|_E \in \text{Aut}_K(E)$. So for every $u \in E \setminus K$ there is an element of $\text{Aut}_K(F)$ which does not fix $u$. So the fixed field of $\text{Aut}_K(F)$ is just $K$; $K = (\text{Aut}_K(F))' = K'$. 
**Lemma V.2.12.** If $F$ is a Galois extension field of $K$ and $E$ is a stable intermediate field of the extension, then $E$ is Galois over $K$.

**Proof.** If $u \in E \setminus K$ then there exists $\sigma \in \text{Aut}_K(F)$ such that $\sigma(u) \neq u$ since $F$ is Galois over $K$ (meaning $K = (\text{Aut}_K(F))' = K'$, so $u \notin K$ implies that $u$ is not fixed by some $\sigma \in \text{Aut}_K(F)$). Since $E$ is stable then $\sigma$ maps $E$ into itself; that is, $\sigma|_E \in \text{Aut}_K(E)$. So for every $u \in E \setminus K$ there is an element of $\text{Aut}_K(F)$ which does not fix $u$. So the fixed field of $\text{Aut}_K(F)$ is just $K$; $K = (\text{Aut}_K(F))' = K'$. Therefore, $E$ is a Galois extension of $K$. \qed
Lemma V.2.12. If $F$ is a Galois extension field of $K$ and $E$ is a stable intermediate field of the extension, then $E$ is Galois over $K$.

Proof. If $u \in E \setminus K$ then there exists $\sigma \in \text{Aut}_K(F)$ such that $\sigma(u) \neq u$ since $F$ is Galois over $K$ (meaning $K = (\text{Aut}_K(F))' = K'$, so $u \notin K$ implies that $u$ is not fixed by some $\sigma \in \text{Aut}_K(F)$). Since $E$ is stable then $\sigma$ maps $E$ into itself; that is, $\sigma|_E \in \text{Aut}_K(E)$. So for every $u \in E \setminus K$ there is an element of $\text{Aut}_K(F)$ which does not fix $u$. So the fixed field of $\text{Aut}_K(F)$ is just $K$; $K = (\text{Aut}_K(F))' = K'$. Therefore, $E$ is a Galois extension of $K$. 

\qed
Lemma V.2.13

Lemma V.2.13. If $F$ is an extension field of $K$ and $E$ is an intermediate field of the extension such that $E$ is algebraic and Galois over $K$, then $E$ is stable (relative to $F$ and $K$).

Proof. If $u \in E$, let $f \in K[x]$ be the irreducible monic polynomial of $u$ and let $u_1, u_2, \ldots, u_r$ be the distinct roots of $f$ that lie in $E$, where $u = u_1$. Then $r \leq n = \deg(f)$ by Theorem III.6.7.
Lemma V.2.13

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Proof. If $u \in E$, let $f \in K[x]$ be the irreducible monic polynomial of $u$ and let $u_1, u_2, \ldots, u_r$ be the distinct roots of $f$ that lie in $E$, where $u = u_1$. Then $r \leq n = \deg(f)$ by Theorem III.6.7. If $\tau \in \text{Aut}_K(E)$, then by Theorem V.2.2 we have that $\tau$ permutes roots of $f$; that is, $\tau$ permutes the $u_i$. Therefore the coefficients of the monic polynomial $g(x) = (x - u_1)(x - u_2) \cdots (x - u_r) \in E[x]$ are fixed by every $\tau \in \text{Aut}_K(E)$, since the coefficients are “symmetric” functions of the $u_i$. Since $E$ is Galois over $K$, then $K = (\text{Aut}_K(E))' = K'$ and so the coefficients are all in $K$ and $g \in K[x]$. 

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Lemma V.2.13. If $F$ is an extension field of $K$ and $E$ is an intermediate field of the extension such that $E$ is algebraic and Galois over $K$, then $E$ is stable (relative to $F$ and $K$).

**Proof.** If $u \in E$, let $f \in K[x]$ be the irreducible monic polynomial of $u$ and let $u_1, u_2, \ldots, u_r$ be the distinct roots of $f$ that lie in $E$, where $u = u_1$. Then $r \leq n = \deg(f)$ by Theorem III.6.7. If $\tau \in \text{Aut}_K(E)$, then by Theorem V.2.2 we have that $\tau$ permutes roots of $f$; that is, $\tau$ permutes the $u_i$. Therefore the coefficients of the monic polynomial 

$$g(x) = (x - u_1)(x - u_2) \cdots (x - u_r) \in E[x]$$

are fixed by every $\tau \in \text{Aut}_K(E)$, since the coefficients are “symmetric” functions of the $u_i$. Since $E$ is Galois over $K$, then $K = (\text{Aut}_K(E))' = K'$ and so the coefficients are all in $K$ and $g \in K[x]$. Now $u = u_1$ is a root of $g$ and hence irreducible $f$ divides $g$ by Theorem V.1.6(ii). Since $g$ is monic and $\deg(g) \leq \deg(f)$ (because $f$ divides $g$) we must have that $f = g$ (or else, $g$ is a divisor of $f$ since $f$ is irreducible). Consequently, all the roots of $f$ are distinct and lie in $E$ (as in the case for $g$).
Lemma V.2.13. If $F$ is an extension field of $K$ and $E$ is an intermediate field of the extension such that $E$ is algebraic and Galois over $K$, then $E$ is stable (relative to $F$ and $K$).

Proof. If $u \in E$, let $f \in K[x]$ be the irreducible monic polynomial of $u$ and let $u_1, u_2, \ldots, u_r$ be the distinct roots of $f$ that lie in $E$, where $u = u_1$. Then $r \leq n = \deg(f)$ by Theorem III.6.7. If $\tau \in \text{Aut}_K(E)$, then by Theorem V.2.2 we have that $\tau$ permutes roots of $f$; that is, $\tau$ permutes the $u_i$. Therefore the coefficients of the monic polynomial $g(x) = (x - u_1)(x - u_2) \cdots (x - u_r) \in E[x]$ are fixed by every $\tau \in \text{Aut}_K(E)$, since the coefficients are “symmetric” functions of the $u_i$. Since $E$ is Galois over $K$, then $K = (\text{Aut}_K(E))' = K'$ and so the coefficients are all in $K$ and $g \in K[x]$. Now $u = u_1$ is a root of $g$ and hence irreducible $f$ divides $g$ by Theorem V.1.6(ii). Since $g$ is monic and $\deg(g) \leq \deg(f)$ (because $f$ divides $g$) we must have that $f = g$ (or else, $g$ is a divisor of $f$ since $f$ is irreducible). Consequently, all the roots of $f$ are distinct and lie in $E$ (as in the case for $g$).
Lemma V.2.13. If $F$ is an extension field of $K$ and $E$ is an intermediate field of the extension such that $E$ is algebraic and Galois over $K$, then $E$ is stable (relative to $F$ and $K$).

**Proof (continued).** Now if $\sigma \in \text{Aut}_K(F)$, then $\sigma(u)$ is a root of $f$ by Theorem V.2.2, whence $\sigma(u) \in E$. Since $u$ was an arbitrary element of $E$, we have shown that $\sigma \in \text{Aut}_K(F)$ maps $E$ into itself; that is, $E$ is stable relative to $K$ and $F$.  \[
\]
Lemma V.2.13. If $F$ is an extension field of $K$ and $E$ is an intermediate field of the extension such that $E$ is algebraic and Galois over $K$, then $E$ is stable (relative to $F$ and $K$).

Proof (continued). Now if $\sigma \in \text{Aut}_K(F)$, then $\sigma(u)$ is a root of $f$ by Theorem V.2.2, whence $\sigma(u) \in E$. Since $u$ was an arbitrary element of $E$, we have shown that $\sigma \in \text{Aut}_K(F)$ maps $E$ into itself; that is, $E$ is stable relative to $K$ and $F$. \hfill \Box
Lemma V.2.14

Lemma V.2.14. Let $F$ be an extension field of $K$ and let $E$ be a stable intermediate field of the extension. Then the quotient group $\text{Aut}_K(F)/\text{Aut}_E(F)$ is isomorphic to the group of all automorphisms in $\text{Aut}_K(E)$ that are extendible to $F$.

Proof. Intermediate field $E$ is stable, so (by the definition of stable) every automorphism $\sigma \in \text{Aut}_K(F)$ maps $E$ into itself, and hence the mapping $\sigma \mapsto \sigma|_E$ defines a group homomorphism from $\text{Aut}_K(F)$ to $\text{Aut}_K(E)$. The image of this homomorphism is “clearly” the subgroup of $\text{Aut}_K(E)$ of all automorphisms that are extendible to $F$ (of course, the extension of $\sigma|_E$ is $\sigma$ itself).
Lemma V.2.14

**Lemma V.2.14.** Let $F$ be an extension field of $K$ and let $E$ be a stable intermediate field of the extension. Then the quotient group $\text{Aut}_K(F)/\text{Aut}_E(F)$ is isomorphic to the group of all automorphisms in $\text{Aut}_K(E)$ that are extendible to $F$.

**Proof.** Intermediate field $E$ is stable, so (by the definition of stable) every automorphism $\sigma \in \text{Aut}_K(F)$ maps $E$ into itself, and hence the mapping $\sigma \mapsto \sigma|_E$ defines a group homomorphism from $\text{Aut}_K(F)$ to $\text{Aut}_K(E)$. The image of this homomorphism is “clearly” the subgroup of $\text{Aut}_K(E)$ of all automorphisms that are extendible to $F$ (of course, the extension of $\sigma|_E$ is $\sigma$ itself). Now the kernel of the homomorphism is all elements of $\text{Aut}_K(F)$ which are the identity on $E$; so the kernel is $\text{Aut}_E(F)$. By the First Isomorphism Theorem (Theorem I.5.7) the homomorphism induces an isomorphism between $\text{Aut}_K(F)$ modulo the kernel and the image of the homomorphism.
Lemma V.2.14. Let $F$ be an extension field of $K$ and let $E$ be a stable intermediate field of the extension. Then the quotient group $\text{Aut}_{K}(F)/\text{Aut}_{E}(F)$ is isomorphic to the group of all automorphisms in $\text{Aut}_{K}(E)$ that are extendible to $F$.

Proof. Intermediate field $E$ is stable, so (by the definition of stable) every automorphism $\sigma \in \text{Aut}_{K}(F)$ maps $E$ into itself, and hence the mapping $\sigma \mapsto \sigma|_{E}$ defines a group homomorphism from $\text{Aut}_{K}(F)$ to $\text{Aut}_{K}(E)$. The image of this homomorphism is “clearly” the subgroup of $\text{Aut}_{K}(E)$ of all automorphisms that are extendible to $F$ (of course, the extension of $\sigma|_{E}$ is $\sigma$ itself). Now the kernel of the homomorphism is all elements of $\text{Aut}_{K}(F)$ which are the identity on $E$; so the kernel is $\text{Aut}_{E}(F)$. By the First Isomorphism Theorem (Theorem I.5.7) the homomorphism induces an isomorphism between $\text{Aut}_{K}(F)$ modulo the kernel and the image of the homomorphism. So $\text{Aut}_{K}(F)/\text{Aut}_{E}(F)$ is isomorphic to the group of all automorphisms in $\text{Aut}_{K}(E)$ that are extendible to $F$. 

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Lemma V.2.14

**Lemma V.2.14.** Let $F$ be an extension field of $K$ and let $E$ be a stable intermediate field of the extension. Then the quotient group $\text{Aut}_K(F)/\text{Aut}_E(F)$ is isomorphic to the group of all automorphisms in $\text{Aut}_K(E)$ that are extendible to $F$.

**Proof.** Intermediate field $E$ is stable, so (by the definition of stable) every automorphism $\sigma \in \text{Aut}_K(F)$ maps $E$ into itself, and hence the mapping $\sigma \mapsto \sigma|_E$ defines a group homomorphism from $\text{Aut}_K(F)$ to $\text{Aut}_K(E)$. The image of this homomorphism is “clearly” the subgroup of $\text{Aut}_K(E)$ of all automorphisms that are extendible to $F$ (of course, the extension of $\sigma|_E$ is $\sigma$ itself). Now the kernel of the homomorphism is all elements of $\text{Aut}_K(F)$ which are the identity on $E$; so the kernel is $\text{Aut}_E(F)$. By the First Isomorphism Theorem (Theorem I.5.7) the homomorphism induces an isomorphism between $\text{Aut}_K(F)$ modulo the kernel and the image of the homomorphism. So $\text{Aut}_K(F)/\text{Aut}_E(F)$ is isomorphic to the group of all automorphisms in $\text{Aut}_K(E)$ that are extendible to $F$.\qed
Theorem V.2.5. The Fundamental Theorem of Galois Theory.

If $F$ is a finite dimensional Galois extension of $K$, then there is a one to one correspondence between the set of all intermediate fields of the extension and the set of all subgroups of the Galois group $\text{Aut}_K(F)$ (given by $E \mapsto E' = \text{Aut}_E(F)$) such that:

(i) the relative dimension of two intermediate fields is equal to the relative index of the corresponding subgroups; in particular, $\text{Aut}_K(F)$ has order $[F : K]$;

(ii) $F$ is Galois over every intermediate field $E$, but $E$ is Galois over $K$ if and only if the corresponding subgroup $E' = \text{Aut}_E(F)$ is normal in $G = \text{Aut}_K(F)$; in the case $G/E'$ is (isomorphic to) the Galois group $\text{Aut}_K(E)$ of $E$ over $K$. 
Theorem V.2.5. The Fundamental Theorem of Galois Theory (i)

**Proof.** Theorem V.2.7 shows that there is a one to one correspondence between the *closed* intermediate fields and *closed* subgroups of the Galois group. By Lemma V.2.10(iii) all intermediate fields are closed and all subgroups of $\text{Aut}_K(F)$ are closed. So the one to one correspondence between closed intermediate fields and closed subgroups is in fact a one to one correspondence between all intermediate fields and all subgroups. This correspondence is given by mapping each group $H$ to its fixed field $H'$ and by mapping each field $M$ to its Galois group $M' = \text{Aut}_M(F)$. 


Theorem V.2.5. The Fundamental Theorem of Galois Theory (i)

Proof. Theorem V.2.7 shows that there is a one to one correspondence between the closed intermediate fields and closed subgroups of the Galois group. By Lemma V.2.10(iii) all intermediate fields are closed and all subgroups of $\text{Aut}_K(F)$ are closed. So the one to one correspondence between closed intermediate fields and closed subgroups is in fact a one to one correspondence between all intermediate fields and all subgroups. This correspondence is given by mapping each group $H$ to its fixed field $H'$ and by mapping each field $M$ to its Galois group $M' = \text{Aut}_M(F)$.

(i) For intermediate fields $L$ and $M$ (with $L \subset M$) we have by Lemma V.2.10(i) that the relative dimension of the fields $[M : L]$ equals the relative index of the corresponding subgroups $[L' : M']$; that is, $[M : L] = [L' : M']$. 
Theorem V.2.5. The Fundamental Theorem of Galois Theory (i)

Proof. Theorem V.2.7 shows that there is a one to one correspondence between the closed intermediate fields and closed subgroups of the Galois group. By Lemma V.2.10(iii) all intermediate fields are closed and all subgroups of \( \text{Aut}_K(F) \) are closed. So the one to one correspondence between closed intermediate fields and closed subgroups is in fact a one to one correspondence between all intermediate fields and all subgroups. This correspondence is given by mapping each group \( H \) to its fixed field \( H' \) and by mapping each field \( M \) to its Galois group \( M' = \text{Aut}_M(F) \).

(i) For intermediate fields \( L \) and \( M \) (with \( L \subset M \)) we have by Lemma V.2.10(i) that the relative dimension of the fields \([M : L]\) equals the relative index of the corresponding subgroups \([L' : M']\); that is, \([M : L] = [L' : M']\). The “in particular” part follows from Lemma V.2.10(iii); that is, \(|\text{Aut}_K(F)| = [F : K]|. \)
Theorem V.2.5. The Fundamental Theorem of Galois Theory (i)

**Proof.** Theorem V.2.7 shows that there is a one to one correspondence between the *closed* intermediate fields and *closed* subgroups of the Galois group. By Lemma V.2.10(iii) all intermediate fields are closed and all subgroups of $\text{Aut}_K(F)$ are closed. So the one to one correspondence between closed intermediate fields and closed subgroups is in fact a one to one correspondence between all intermediate fields and all subgroups. This correspondence is given by mapping each group $H$ to its fixed field $H'$ and by mapping each field $M$ to its Galois group $M' = \text{Aut}_M(F)$.

(i) For intermediate fields $L$ and $M$ (with $L \subset M$) we have by Lemma V.2.10(i) that the relative dimension of the fields $[M : L]$ equals the relative index of the corresponding subgroups $[L' : M']$; that is, $[M : L] = [L' : M']$. The “in particular” part follows from Lemma V.2.10(iii); that is, $|\text{Aut}_K(F)| = [F : K]$. 
Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii)

(ii) $F$ is Galois over every intermediate field $E$, but $E$ is Galois over $K$ if and only if the corresponding subgroup $E' = \text{Aut}_E(F)$ is normal in $G = \text{Aut}_K(F)$; in the case $G/E'$ is (isomorphic to) the Galois group $\text{Aut}_K(E)$ of $E$ over $K$.

Proof. $F$ is Galois over $E$ since $E$ is closed (see the comment after the definition of closed), so $F$ is Galois over every intermediate field. $E$ is finite dimensional over $K$ (since $F$ is; see Theorem V.1.2) and hence, by Theorem V.1.11, $F$ is algebraic over $K$. Consequently if $E$ is Galois over $K$ then $E$ satisfies the hypotheses of Lemma V.2.13 and so $E$ is stable relative to $F$ and $K$. By Lemma V.2.11(i), $E' = \text{Aut}_E(F)$ is normal in $G = \text{Aut}_K(F)$ (this is the first part of the claim of (ii)).
Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii)

(ii) $F$ is Galois over every intermediate field $E$, but $E$ is Galois over $K$ if and only if the corresponding subgroup $E' = \text{Aut}_E(F)$ is normal in $G = \text{Aut}_K(F)$; in the case $G/E'$ is (isomorphic to) the Galois group $\text{Aut}_K(E)$ of $E$ over $K$.

Proof. $F$ is Galois over $E$ since $E$ is closed (see the comment after the definition of closed), so $F$ is Galois over every intermediate field. $E$ is finite dimensional over $K$ (since $F$ is; see Theorem V.1.2) and hence, by Theorem V.1.11, $F$ is algebraic over $K$. Consequently if $E$ is Galois over $K$ then $E$ satisfies the hypotheses of Lemma V.2.13 and so $E$ is stable relative to $F$ and $K$. By Lemma V.2.11(i), $E' = \text{Aut}_E(F)$ is normal in $G = \text{Aut}_K(F)$ (this is the first part of the claim of (ii)). Conversely, if $E' = \text{Aut}_E(F)$ is normal in $G = \text{Aut}_K(F)$, then by Lemma V.2.11(ii), $E''$ is a stable intermediate field.
Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii)

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Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii) (continued)

Proof (continued). We establish that all intermediate fields are closed, so $E = E''$ and $E$ is stable. Therefore by Lemma V.2.12, $E$ is Galois over $K$ (this is the second part of the claim of (ii), the converse of the first part).

Now for the “in the case” part.
Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii) (continued)

**Proof (continued).** We establish that all intermediate fields are closed, so \( E = E'' \) and \( E \) is stable. Therefore by Lemma V.2.12, \( E \) is Galois over \( K \) (this is the second part of the claim of (ii), the converse of the first part).

Now for the “in the case” part. Let \( E' = \text{Aut}_E(F) \) be normal in \( G = \text{Aut}_K(F) \), or equivalently, let \( E \) be Galois over \( K \). We have seen at the beginning of the proof that all intermediate fields and subgroups are closed, so \( E \) and \( E' \) are closed. Since \( F \) is Galois over \( K \) then \( G' = (\text{Aut}_K(F))' = K \). Now the elements of \( G/E' \) are cosets of \( E' \), so \( |G/E'| = [G : E'] \).
Proof (continued). We establish that all intermediate fields are closed, so $E = E''$ and $E$ is stable. Therefore by Lemma V.2.12, $E$ is Galois over $K$ (this is the second part of the claim of (ii), the converse of the first part).

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|G/E'| = [G : E'] \\
= [E'' : G'] \text{ by Lemma V.2.10}(ii) \\
= [E : K] \text{ since } E = E'' \text{ and } K = G'.
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Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii) (continued)

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Proof. We saw above that $E''$ is stable and $E = E''$, so $E$ is stable. By Lemma V.2.14, $G/E' = \text{Aut}_K(F)/\text{Aut}_E(F)$ is isomorphic to a subgroup of $\text{Aut}_K(E)$. Since we have just shown that $|G/E'| = [E : K]$, then this subgroup of $\text{Aut}_K(E)$ is of order $[E : K]$. Since $E$ is Galois over $K$ (by hypothesis, here) then part (i) shows that $|\text{Aut}_K(E)| = [E : K]$. 
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\qed
Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii)

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Theorem V.2.15. (Artin.)

Let $F$ be a field, $G$ a group of automorphisms of $F$, and $K$ the fixed field of $G$ in $F$. Then $F$ is Galois over $K$. If $G$ is finite, then $F$ is a finite dimensional Galois extension of $K$ with Galois group $G$.

Proof. Since $K$ is the fixed field of $G$ in $F$, then for each $u \in F \setminus K$ there must be a $\sigma \in G$ such that $\sigma(u) \neq u$. 
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Proof. Since $K$ is the fixed field of $G$ in $F$, then for each $u \in F \setminus K$ there must be a $\sigma \in G$ such that $\sigma(u) \neq u$. By the definition of $G$ as a group of automorphisms of $F$ which fixes $K$ elementwise, we have $G < \text{Aut}_K(F)$ (since $\text{Aut}_K(F)$ fixes $K$ elementwise, as well as possibly other things). So each such $\sigma \in G$ is also in $\text{Aut}_K(F)$ and therefore the fixed field of $\text{Aut}_K(F)$ is $K$ itself.
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Proof (continued). If $G$ is finite, then by Lemma V.2.9 (with $H = 1$ and $J = G$, which gives $|G| = [G : 1]$ is finite) we have

$$[F : K] = [1' : G'] \text{ since } 1' = F \text{ and } G' = K$$

(fixed fields of 1 and $G$, respectively)

$$\leq [G : 1] \text{ by Lemma V.2.9}$$

$$= |G|.$$

Consequently, $F$ is finite dimensional over $K$. So $F$ is a finite dimensional Galois extension of $K$, and so by Lemma V.2.10(iii) all intermediate groups are closed and so $G = G''$. Since the fixed field of $G$ is $G' = K$ (and hence $G'' = K'$) we have that the Galois group of $F$ over $K$ is

$$\text{Aut}_K(F) = K' \text{ by the prime notation}$$

$$= G'' \text{ as just observed}$$

$$= G \text{ since } G \text{ is closed.}$$
Theorem V.2.15. Artin (continued)

Proof (continued). If $G$ is finite, then by Lemma V.2.9 (with $H = 1$ and $J = G$, which gives $|G| = [G : 1]$ is finite) we have

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