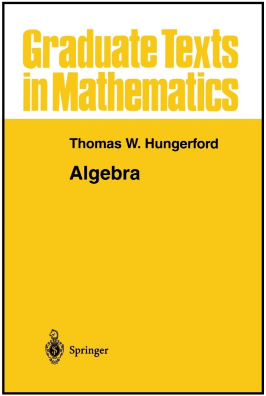
Modern Algebra

Chapter V. Fields and Galois Theory

V.2. The Fundamental Theorem (of Galois Theory)—Proofs of Theorems



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Proof. Let $f = \sum_{i=0}^{n} k_i x^i$. Since σ fixes K, $\sigma(0) = 0$ and so $f(u) = 0$ implies

$$0 = \sigma(0) = \sigma(f(u)) = \sigma\left(\sum_{i=0}^{n} k_i x^i\right)$$

$$= \sum_{i=0}^{n} \sigma(k_i)\sigma(u^i) = \sum_{i=0}^{n} k_i(\sigma(u))^i = f(\sigma(u)).$$

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Lemma V.2.6

Lemma V.2.6. Let F be an extension field of K with intermediate fields L and M (say $K \subset L \subset M \subset F$). Let H and J be subgroups of $G = \mathrm{Aut}_K(F)$. Then:

(i) $F' = 1$ (the identity group) and $K' = G$;

(i′) $1' = F$;

(ii) $L \subset M$ implies $M' < L'$;

(ii′) $H < J$ implies $J' \subset H'$;

(iii) $L \subset L''$ and $H < H''$ (where $L'' = (L')'$ and $H'' = (H')'$);

(iv) $L' = L'''$ and $H' = H'''$.

Proof. **(i)** Now $F' = \mathrm{Aut}_F(F)$ is the group of automorphisms of F which fix F and hence must consist only of the identity permutation and so F' is the "identity group."

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(i′) $1'$ is the fixed field of the identity group. F is the "universal field" and the identity group fixes all of F; i.e., $1' = F$.

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(ii′) Suppose subgroups H, J of $G = \text{Aut}_K(F)$ satisfy $H < J$. Now an element of J' (the fixed field of J) is fixed by every element of J and, since $H < J$, also fixed by every element of H.

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Proof. (iii) Let L be an intermediate field. Then $L' = \mathrm{Aut}_L(F)$ is a group, and L'' is the fixed field of L'. Now any element of L is fixed by $L' = \mathrm{Aut}_L(F)$.

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Proof. (iii) Let L be an intermediate field. Then $L' = \text{Aut}_L(F)$ is a group, and L'' is the fixed field of L'. Now any element of L is fixed by $L' = \text{Aut}_L(F)$. Also, L'' includes everything in F fixed by the elements of $L' = \text{Aut}_L(F)$. So L'' includes all of L (and possibly more); $L \subset L''$.

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Let H by a subgroup of $G = \text{Aut}_K(F)$. Then H' is the fixed field of H. Now $(H')' = H''$ is the group of permutations of F which fix H'.

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(iv) $L' = L'''$ and $H' = H'''$.

Proof. (iv) Let L be an intermediate field. By (iii), $L \subset L''$ and so by (ii), $L''' < L'$. Now L' is a subgroup of $G = \mathrm{Aut}_K(F)$ and so by (iii) (with H replaced with L') we have $L' < L'''$, and so $L' = L'''$.

Let H be a subgroup of $G = \mathrm{Aut}_K(F)$. By (iii), $H < H''$ and so by (ii′), $H''' \subset H'$. Now H' is an intermediate field so by (iii) (with L replaced with H') we have $H' \subset H'''$, and so $H' = H'''$. □

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Lemma V.2.8

Lemma V.2.8. Let F be an extension field of K and L, M intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

Proof. (Notice that $[M : L]$ and $[F : K]$ are dimensions of vector spaces; $[L' : M']$, the index of L' over M', is the number of cosets of L in M.)

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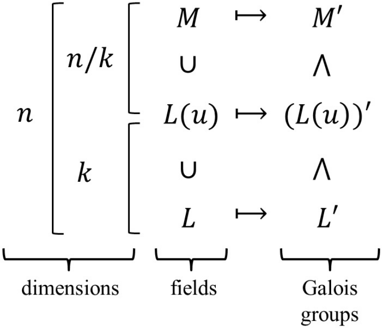
Proof. (Notice that $[M : L]$ and $[F : K]$ are dimensions of vector spaces; $[L' : M']$, the index of L' over M', is the number of cosets of L in M.) Since $[M : L]$ is finite, we give a proof based on induction. Let $n = [M : L]$. If $n = 1$ then $M + L$ and so $M' = L'$ and $[L' : M'] = 1$, so the result holds. Let $n > 1$ and suppose the theorem holds for all $i < n$. Since $n > 1$, there is some $u \in M$ with $u \notin L$. Since $[M : L]$ is finite, then u is algebraic over L by Theorem V.1.11. Let $f \in L[x]$ be the irreducible monic polynomial of u, say of degree $k > 1$. By Theorem V.1.6(iii), $[L(u) : L] = k$. By Theorem V.1.1, $[M : L] = [M : L(u)][L(u) : L]$ and so $[M : L(u)] = n/k$.

Lemma V.2.8 (continued 1)

Proof (continued). Schematically:



We now consider two cases.

Case 1. If $k < n$ then $1 < n/k < n$; and

Case 2. If $k = n$.

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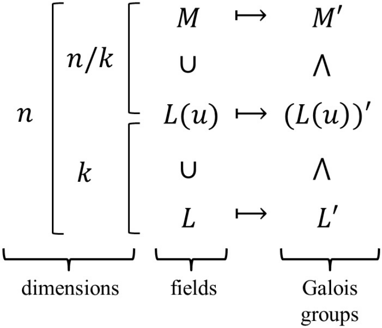

We now consider two cases.

<u>Case 1.</u> If $k < n$ then $1 < n/k < n$; and

<u>Case 2.</u> If $k = n$.

Lemma V.2.8 (continued 2)

Lemma V.2.8. Let F be an extension field of K and L, M intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\text{Aut}_K(F)| \leq [F : K]$.

Proof (continued).
Case 1. If $k < n$ then $1 < n/k < n$. By the induction hypothesis, since $i = n/k < n$, we have that $L \subset L(u)$ implies $[L' : (L(u))'] \leq [L(u) : L] = k$, and that $L(U) \subset M$ implies $[(L(u))' : M'] \leq [M : L(u)] = n/k$.

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Case 1. If $k < n$ then $1 < n/k < n$. By the induction hypothesis, since $i = n/k < n$, we have that $L \subset L(u)$ implies $[L' : (L(u))'] \leq [L(u) : L] = k$, and that $L(U) \subset M$ implies $[(L(u))' : M'] \leq [M : L(u)] = n/k$. Hence

$$\begin{aligned} [L' : M'] &= [L' : (L(u))'][(L(u))' : M'] \text{ by Theorem V.1.1} \\ &\leq k(n/k) = n = [M : L] \end{aligned}$$

and the theorem holds in this case.

Lemma V.2.8 (continued 2)

Lemma V.2.8. Let F be an extension field of K and L, M intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\mathrm{Aut}_K(F)| \leq [F : K]$.

Proof (continued).
Case 1. If $k < n$ then $1 < n/k < n$. By the induction hypothesis, since $i = n/k < n$, we have that $L \subset L(u)$ implies $[L' : (L(u))'] \leq [L(u) : L] = k$, and that $L(U) \subset M$ implies $[(L(u))' : M'] \leq [M : L(u)] = n/k$. Hence

$$\begin{aligned} [L' : M'] &= [L' : (L(u))'][(L(u))' : M'] \text{ by Theorem V.1.1} \\ &\leq k(n/k) = n = [M : L] \end{aligned}$$

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Lemma V.2.8 (continued 3)

Lemma V.2.8. Let F be an extension field of K and L, M intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\mathrm{Aut}_K(F)| \leq [F : K]$.

Proof (continued).
Case 2. On the other hand, if $k = n$ then by Theorem V.1.1, $[M : L] = [M : L(u)][L(u) : L]$ and so $[M" L(u)] = 1$ (as above). So $M = L(u)$.

Lemma V.2.8 (continued 3)

Lemma V.2.8. Let F be an extension field of K and L, M intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\mathrm{Aut}_K(F)| \leq [F : K]$.

Proof (continued).
Case 2. On the other hand, if $k = n$ then by Theorem V.1.1, $[M : L] = [M : L(u)][L(u) : L]$ and so $[M" L(u)] = 1$ (as above). So $M = L(u)$. In the final part of the proof, we will construct an injective map from the set S of all left costs of M' in L' (of which there are $[L' : M']$ such cosets) to the set T of all distinct roots in F of the polynomial $f \in L[x]$ (of which there are at most $k \leq n$ such roots by Theorem III.6.7).

Lemma V.2.8 (continued 3)

Lemma V.2.8. Let F be an extension field of K and L, M intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\mathrm{Aut}_K(F)| \leq [F : K]$.

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Lemma V.2.8 (continued 3)

Lemma V.2.8. Let F be an extension field of K and L, M intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\mathrm{Aut}_K(F)| \leq [F : K]$.

Proof (continued).
Case 2. On the other hand, if $k = n$ then by Theorem V.1.1, $[M : L] = [M : L(u)][L(u) : L]$ and so $[M" L(u)] = 1$ (as above). So $M = L(u)$. In the final part of the proof, we will construct an injective map from the set S of all left costs of M' in L' (of which there are $[L' : M']$ such cosets) to the set T of all distinct roots in F of the polynomial $f \in L[x]$ (of which there are at most $k \leq n$ such roots by Theorem III.6.7). So we have $|S| = [L' : M']$ and $|T| \leq n$, the existence of the injective map from S to T gives that $|S| \leq |T|$ and it will then follow that $[L' : M'] = |S| \leq |T| \leq n = [M : L]$, establishing the theorem in this second case.

Lemma V.2.8 (continued 4)

Proof (continued). Now for the construction of the injective map from S to T. Let $\tau \in L'$ and $\tau M'$ a left coset of M' in L'. If $\sigma \in M' = \text{Aut}_M(F)$, then since $u \in M$ (by choice, above) we have that $\sigma(u) = u$ and so $\tau\sigma(u) = \tau(u)$; so every element of the coset $\tau M'$ (this is a group element which acts on elements of F, u in particular) has the same effect on u and maps $u \mapsto \tau(u)$ (that is, there is independence of element $\sigma \in M'$).

Lemma V.2.8 (continued 4)

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Lemma V.2.8 (continued 4)

Proof (continued). Now for the construction of the injective map from S to T. Let $\tau \in L'$ and $\tau M'$ a left coset of M' in L'. If $\sigma \in M' = \mathrm{Aut}_M(F)$, then since $u \in M$ (by choice, above) we have that $\sigma(u) = u$ and so $\tau\sigma(u) = \tau(u)$; so every element of the coset $\tau M'$ (this is a group element which acts on elements of F, u in particular) has the same effect on u and maps $u \mapsto \tau(u)$ (that is, there is independence of element $\sigma \in M'$). Since $\tau \in L' = \mathrm{Aut}_L(F)$ (because $\tau M'$ is a coset in L') and u is a root of $f \in L[x]$, then $\tau(u)$ is also a root of f by Theorem V.2.2. This implies that the map $S \mapsto T$ given by $\tau M' \mapsto \tau(u)$ is well-defined (HMMMM; that is, the mapping actually produces an element of T, the set of roots of f). If $\tau(u) = \tau_0(u)$ for $\tau, \tau_0 \in L'$ then $\tau_0^{-1}\tau(u) = u$ (L' is a group of permutations, so inverses exist) and hence $\tau_0\tau$ fixes u. Since $\tau, \tau_0 \in L' = \mathrm{Aut}_L(F)$ then certainly τ, τ_0, and τ_0^{-1} fixes L, so $\tau_o^{-1}\tau$ fixes $L(u) = M$ elementwise (recall that a basis for $L(u) = M$ over L is $\{1, u, u^2, \ldots, u^{k-1}\}$ by Theorem V.1.6(iv)) and $\tau_0\tau \in M'$.

Lemma V.2.8 (continued 5)

Lemma V.2.8. Let F be an extension field of K and L, M intermediate fields with $L \subset M$. If $M : L$ is finite, then $[L' : M'] \leq [M : L]$. In particular, if $[F : K]$ is finite, then $|\mathrm{Aut}_K(F)| \leq [F : K]$.

Proof (continued). Consequently by Corollary V.4.3(iii), $\tau_0 M' = \tau M'$ and so the map $S \to T$ is one to one (injective) and this completes the second case of the induction. Therefore $[L' : M'] \leq [M : L]$.

For the "in particular" part of the proof, notice that $\mathrm{Aut}_K(F) \cong \mathrm{Aut}_K(F)/1$ (where "1" is the trivial "identity group"). So $|\mathrm{Aut}_K(F)| = [\mathrm{Aut}_K(F) : 1]$.

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Lemma V.2.8 (continued 5)

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For the "in particular" part of the proof, notice that $\mathrm{Aut}_K(F) \cong \mathrm{Aut}_K(F)/1$ (where "1" is the trivial "identity group"). So $|\mathrm{Aut}_K(F)| = [\mathrm{Aut}_K(F) : 1]$. Also, in the prime notation $K' = \mathrm{Aut}_K(F)$ and $F' = \mathrm{Aut}_F(F) = 1$, so $|\mathrm{Aut}_K(F)| = [\mathrm{Aut}_K(F) : 1] = [K' : F'] \leq [F : K]$ with $L = K$ and $M = F$, from the above result. □

Lemma V.2.9

Lemma V.2.9. Let F be an extension field of K and let H, J be subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$. If $[J : H]$ is finite, then $[H' : J'] \leq [J : H]$.

Proof. (Notice that $[H' : J']$ is the dimension of field H' as a vector space over field J'; the index $[J : H]$ is the number of cosets of H in J.) Let the number of cosets of H in J by $[J : H] = n$.

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Proof. (Notice that $[H' : J']$ is the dimension of field H' as a vector space over field J'; the index $[J : H]$ is the number of cosets of H in J.) Let the number of cosets of H in J by $[J : H] = n$. ASSUME $[H' : J'] > n$. Then a basis of H' over J' has more than n elements (as basis is a linearly independent spanning set; see page 181) and so there exist $u_1, u_2, \ldots, u_{n+1} \in H'$ that are linearly independent over J'.

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Lemma V.2.9 (continued 1)

$$
\begin{aligned}
\tau_1(u_1)x_1 + \tau_1(u_2)x_2 + \tau_1(u_3)x_3 + \cdots + \tau_1(u_{n+1})x_{n+1} &= 0 \\
\tau_2(u_1)x_1 + \tau_2(u_2)x_2 + \tau_2(u_3)x_3 + \cdots + \tau_2(u_{n+1})x_{n+1} &= 0 \\
\tau_3(u_1)x_1 + \tau_3(u_2)x_2 + \tau_3(u_3)x_3 + \cdots + \tau_3(u_{n+1})x_{n+1} &= 0 \\
&\vdots \qquad (1) \\
\tau_n(u_1)x_1 + \tau_n(u_2)x_2 + \tau_n(u_3)x_3 + \cdots + \tau_n(u_{n+1})x_{n+1} &= 0.
\end{aligned}
$$

Such a system (n homogeneous equations in $n+1$ unknowns) has a nontrivial solution as will be shown in Exercise VII.2.4(d) (see also Lemma 5.1.1 of *Real Analysis with an Introduction to Wavelets*, Don Hong, Jianzhong Wang, and Robert Gardner, Academic Press/Elsevier Press, 2005).

Lemma V.2.9 (continued 1)

$$\begin{aligned}
\tau_1(u_1)x_1 + \tau_1(u_2)x_2 + \tau_1(u_3)x_3 + \cdots + \tau_1(u_{n+1})x_{n+1} &= 0\\
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Lemma V.2.9 (continued 1)

$$\begin{aligned}
\tau_1(u_1)x_1 + \tau_1(u_2)x_2 + \tau_1(u_3)x_3 + \cdots + \tau_1(u_{n+1})x_{n+1} &= 0 \\
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&\vdots \qquad (1) \\
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\end{aligned}$$

Such a system (n homogeneous equations in $n+1$ unknowns) has a nontrivial solution as will be shown in Exercise VII.2.4(d) (see also Lemma 5.1.1 of *Real Analysis with an Introduction to Wavelets*, Don Hong, Jianzhong Wang, and Robert Gardner, Academic Press/Elsevier Press, 2005). Among all such nontrivial solutions choose one, say $x_1 = a_1, x_2 = a_2, \ldots, x_{n+1} = a_{n+1}$ with a minimal number of nonzero a_i. By reindexing if necessary we may assume that $x_1 = a_1$, $x_2 = a_2$, $\ldots$, $x_r = a_r$ and $x_{r+1} = x_{r+1} = \cdots x_{n+1} = 0$ where $a_r \neq 0$. Since each multiple of a solution is also a solution then we may also assume $a_1 = 1_F$.

Lemma V.2.9 (continued 2)

Proof (continued). In the conclusion of the proof below, we will show that the hypothesis that $u_1, u_2, \ldots, u_{n+1} \in H'$ are linearly independent over J' implies that there exists $\sigma \in J$ such that $x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r$ and $x_{r+1} = x_{r+2} = \cdots = x_{n+1} = 0$ is also a nontrivial solution to the system of equations (1) and $\sigma a_2 = a_2$. Since the difference of two solutions is also a solution (since the system (1) is linear and homogeneous) then

$$x_1 = a_1 - \sigma a_1, x_2 = a_2 - \sigma a_2, \ldots, x_t = a_r - \sigma a_r, \text{ and}$$
$$x_{r+1} = x_{r+2} = \cdots = x_{n+1} = 0 \qquad (*)$$

is also a solution of the system of equations (1).

Lemma V.2.9 (continued 2)

Proof (continued). In the conclusion of the proof below, we will show that the hypothesis that $u_1, u_2, \ldots, u_{n+1} \in H'$ are linearly independent over J' implies that there exists $\sigma \in J$ such that $x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r$ and $x_{r+1} = x_{r+2} = \cdots = x_{n+1} = 0$ is also a nontrivial solution to the system of equations (1) and $\sigma a_2 = a_2$. Since the difference of two solutions is also a solution (since the system (1) is linear and homogeneous) then

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Lemma V.2.9 (continued 2)

Proof (continued). In the conclusion of the proof below, we will show that the hypothesis that $u_1, u_2, \ldots, u_{n+1} \in H'$ are linearly independent over J' implies that there exists $\sigma \in J$ such that $x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r$ and $x_{r+1} = x_{r+2} = \cdots = x_{n+1} = 0$ is also a nontrivial solution to the system of equations (1) and $\sigma a_2 = a_2$. Since the difference of two solutions is also a solution (since the system (1) is linear and homogeneous) then

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Lemma V.2.9 (continued 3)

Proof (continued). This contradiction shows that the assumption $[H' : J'] > n$ is false, and hence $[H' : J'] \leq n$.

To complete the proof, we must find $\sigma \in J$ with the desired properties. Now $\{\tau_1, \tau_2, \ldots, \tau_n\}$ is a set of representatives of the cosets of H, then exactly one of the τ_j, say τ_1, is in H itself.

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Proof (continued). This contradiction shows that the assumption $[H' : J'] > n$ is false, and hence $[H' : J'] \leq n$.

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Proof (continued). This contradiction shows that the assumption $[H' : J'] > n$ is false, and hence $[H' : J'] \leq n$.

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Lemma V.2.9 (continued 4)

Proof (continued). Next, consider a second system of equations (which we will show to be equivalent to [that is, have the identical solutions as] the first system of equations (1)):

$$\begin{aligned}
\sigma\tau_1(u_1)x_1 + \sigma\tau_1(u_2)x_2 + \sigma\tau_1(u_3)x_3 + \cdots + \sigma\tau_1(u_{n+1})x_{n+1} &= 0 \\
\sigma\tau_2(u_1)x_1 + \sigma\tau_2(u_2)x_2 + \sigma\tau_2(u_3)x_3 + \cdots + \sigma\tau_2(u_{n+1})x_{n+1} &= 0 \\
\sigma\tau_3(u_1)x_1 + \sigma\tau_3(u_2)x_2 + \sigma\tau_3(u_3)x_3 + \cdots + \sigma\tau_3(u_{n+1})x_{n+1} &= 0 \\
&\vdots \qquad (2) \\
\sigma\tau_n(u_1)x_1 + \sigma\tau_n(u_2)x_2 + \sigma\tau_n(u_3)x_3 + \cdots + \sigma\tau_n(u_{n+1})x_{n+1} &= 0.
\end{aligned}$$

Since $\sigma \in J < \mathrm{Aut}_K(F)$ then $\sigma(0) = 0$ and if we apply σ to each of the equations in the first system (1), then we get the second system (2). Since $x_1 = 1_1, x_2 = a_2, \ldots, x_r = a_r$ and $x_{r+1} = x_{r+2} = \cdots x_{n+1} = 0$ is a solution of system (1), then $x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r$ and $x_{r+1} = x_{r+2} = \cdots = x_{n+1} = 0$ is a solution of system (2).

Lemma V.2.9 (continued 4)

Proof (continued). Next, consider a second system of equations (which we will show to be equivalent to [that is, have the identical solutions as] the first system of equations (1)):

$$\begin{aligned}
\sigma\tau_1(u_1)x_1 + \sigma\tau_1(u_2)x_2 + \sigma\tau_1(u_3)x_3 + \cdots + \sigma\tau_1(u_{n+1})x_{n+1} &= 0 \\
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\sigma\tau_3(u_1)x_1 + \sigma\tau_3(u_2)x_2 + \sigma\tau_3(u_3)x_3 + \cdots + \sigma\tau_3(u_{n+1})x_{n+1} &= 0 \\
&\vdots \qquad (2) \\
\sigma\tau_n(u_1)x_1 + \sigma\tau_n(u_2)x_2 + \sigma\tau_n(u_3)x_3 + \cdots + \sigma\tau_n(u_{n+1})x_{n+1} &= 0.
\end{aligned}$$

Since $\sigma \in J < \mathrm{Aut}_K(F)$ then $\sigma(0) = 0$ and if we apply σ to each of the equations in the first system (1), then we get the second system (2). Since $x_1 = 1_1, x_2 = a_2, \ldots, x_r = a_r$ and $x_{r+1} = x_{r+2} = \cdots x_{n+1} = 0$ is a solution of system (1), then $x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r$ and $x_{r+1} = x_{r+2} = \cdots = x_{n+1} = 0$ is a solution of system (2).

Lemma V.2.9 (continued 5)

Proof (continued). We claim that system (2), except for the order of the equations, is identical with system (1) (so that $x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r$ and $x_{r+1} = x_{r+1} = \cdots = x_{n+1} = 0$ is a solution of system (1); this will show that σ satisfies the conditions mentioned above). We make two claims:

(i) For any $\sigma \in J$, the set $\{\sigma\tau_1, \sigma\tau_2, \ldots, \sigma\tau_n\} \subset J$ is a complete set of coset representatives of the cosets of H in J.

Lemma V.2.9 (continued 5)

Proof (continued). We claim that system (2), except for the order of the equations, is identical with system (1) (so that $x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r$ and $x_{r+1} = x_{r+1} = \cdots = x_{n+1} = 0$ is a solution of system (1); this will show that σ satisfies the conditions mentioned above). We make two claims:

(i) For any $\sigma \in J$, the set $\{\sigma\tau_1, \sigma\tau_2, \ldots, \sigma\tau_n\} \subset J$ is a complete set of coset representatives of the cosets of H in J.

Sub-Proof. First, since each $\tau_i \in J$ and $\sigma \in J$, then $\sigma\tau_i \in J$. Now $\sigma\tau_i H = \sigma\tau_i H$ if and only if $(\sigma\tau_i)^{-1}(\sigma\tau_j) \in H$ by Theorem I.4.3(iii); that is, $\tau_i^{-1}\sigma^{-1}\sigma\tau_j = \tau_i^{-1}\tau_j \in H$.

Lemma V.2.9 (continued 5)

Proof (continued). We claim that system (2), except for the order of the equations, is identical with system (1) (so that $x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r$ and $x_{r+1} = x_{r+1} = \cdots = x_{n+1} = 0$ is a solution of system (1); this will show that σ satisfies the conditions mentioned above). We make two claims:

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Sub-Proof. First, since each $\tau_i \in J$ and $\sigma \in J$, then $\sigma\tau_i \in J$. Now $\sigma\tau_i H = \sigma\tau_i H$ if and only if $(\sigma\tau_i)^{-1}(\sigma\tau_j) \in H$ by Theorem I.4.3(iii); that is, $\tau_i^{-1}\sigma^{-1}\sigma\tau_j = \tau_i^{-1}\tau_j \in H$. Again by Theorem I.4.3(iii), $\tau_i^{-1}\tau_j \in H$ if only if $\tau_i H = \tau_j H$. So $\sigma\tau_i H = \sigma\tau_j H$ if and only if $\tau_i H = \tau_j H$. Since $\{\tau_1, \tau_2, \ldots, \tau_n\}$ is a complete set of representatives of the left cosets of H in J, then so is $\{\sigma\tau_1, \sigma\tau_2, \ldots, \sigma\tau_n\}$. *Sub-Q.E.D.*

Lemma V.2.9 (continued 5)

Proof (continued). We claim that system (2), except for the order of the equations, is identical with system (1) (so that $x_1 = \sigma a_1, x_2 = \sigma a_2, \ldots, x_r = \sigma a_r$ and $x_{r+1} = x_{r+1} = \cdots = x_{n+1} = 0$ is a solution of system (1); this will show that σ satisfies the conditions mentioned above). We make two claims:

(i) For any $\sigma \in J$, the set $\{\sigma\tau_1, \sigma\tau_2, \ldots, \sigma\tau_n\} \subset J$ is a complete set of coset representatives of the cosets of H in J.

<u>Sub-Proof.</u> First, since each $\tau_i \in J$ and $\sigma \in J$, then $\sigma\tau_i \in J$. Now $\sigma\tau_i H = \sigma\tau_i H$ if and only if $(\sigma\tau_i)^{-1}(\sigma\tau_j) \in H$ by Theorem I.4.3(iii); that is, $\tau_i^{-1}\sigma^{-1}\sigma\tau_j = \tau_i^{-1}\tau_j \in H$. Again by Theorem I.4.3(iii), $\tau_i^{-1}\tau_j \in H$ if only if $\tau_i H = \tau_j H$. So $\sigma\tau_i H = \sigma\tau_j H$ if and only if $\tau_i H = \tau_j H$. Since $\{\tau_1, \tau_2, \ldots, \tau_n\}$ is a complete set of representatives of the left cosets of H in J, then so is $\{\sigma\tau_1, \sigma\tau_2, \ldots, \sigma\tau_n\}$. *Sub-Q.E.D.*

Lemma V.2.9 (continued 6)

Proof (continued).
(ii) If ζ and θ are both elements in the same coset of H in J, then (since $u_i \in H'$) $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots n + 1$.

Sub-Proof. Let $\zeta, \theta \in aH$. Then $\zeta = ah_1$ and $\theta = ah_2$ for some $h_1, h_2 \in H$.

Lemma V.2.9 (continued 6)

Proof (continued).
(ii) If ζ and θ are both elements in the same coset of H in J, then (since $u_i \in H'$) $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots n + 1$.

Sub-Proof. Let $\zeta, \theta \in aH$. Then $\zeta = ah_1$ and $\theta = ah_2$ for some $h_1, h_2 \in H$. Since H' is the fixed field of H and each $u_i \in H'$, then $\zeta(u_i) = (ah_1)(u_i) = ah_1(u_i) = au_i$ and $\theta(u_i) = (ah_2)(u_i) = ah_2(u_i) = au_i$. So $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots, n + 1$. *Sub-Q.E.D.*

Lemma V.2.9 (continued 6)

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(ii) If ζ and θ are both elements in the same coset of H in J, then (since $u_i \in H'$) $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots n+1$.

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It now follows from claim (i) that there is some reordering $i_1, i_2, \ldots, i_{n+1}$ of $1, 2, \ldots, n+1$ so that for each $k = 1, 2, \ldots, n+1$, $\sigma\tau_k$ and τ_{i_k} are in the same coset of H in J. By (ii), the kth equation of system (2) (with coefficients $\sigma\tau_k(u_i)$) is identical with the i_kth equation of system (1) (with coefficients $\tau_{i_k}(u_i)$).

Lemma V.2.9 (continued 6)

Proof (continued).
(ii) If ζ and θ are both elements in the same coset of H in J, then (since $u_i \in H'$) $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots n+1$.

Sub-Proof. Let $\zeta, \theta \in aH$. Then $\zeta = ah_1$ and $\theta = ah_2$ for some $h_1, h_2 \in H$. Since H' is the fixed field of H and each $u_i \in H'$, then $\zeta(u_i) = (ah_1)(u_i) = ah_1(u_i) = au_i$ and $\theta(u_i) = (ah_2)(u_i) = ah_2(u_i) = au_i$. So $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots, n+1$. *Sub-Q.E.D.*

It now follows from claim (i) that there is some reordering $i_1, i_2, \ldots, i_{n+1}$ of $1, 2, \ldots, n+1$ so that for each $k = 1, 2, \ldots, n+1$, $\sigma\tau_k$ and τ_{i_k} are in the same coset of H in J. By (ii), the kth equation of system (2) (with coefficients $\sigma\tau_k(u_i)$) is identical with the i_kth equation of system (1) (with coefficients $\tau_{i_k}(u_i)$). So we have, in particular, that the solution $x_1 = a_1, x_2 = a_2, \ldots, x_r = a_r$ and $x_{r+1} = x_{r+2} = \cdots x_{n+1} = 0$ of system (2) is also a solution of system (1).

Lemma V.2.9 (continued 6)

Proof (continued).
(ii) If ζ and θ are both elements in the same coset of H in J, then (since $u_i \in H'$) $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots n + 1$.

Sub-Proof. Let $\zeta, \theta \in aH$. Then $\zeta = ah_1$ and $\theta = ah_2$ for some $h_1, h_2 \in H$. Since H' is the fixed field of H and each $u_i \in H'$, then $\zeta(u_i) = (ah_1)(u_i) = ah_1(u_i) = au_i$ and $\theta(u_i) = (ah_2)(u_i) = ah_2(u_i) = au_i$. So $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots, n + 1$. *Sub-Q.E.D.*

It now follows from claim (i) that there is some reordering $i_1, i_2, \ldots, i_{n+1}$ of $1, 2, \ldots, n + 1$ so that for each $k = 1, 2, \ldots, n + 1$, $\sigma\tau_k$ and τ_{i_k} are in the same coset of H in J. By (ii), the kth equation of system (2) (with coefficients $\sigma\tau_k(u_i)$) is identical with the i_kth equation of system (1) (with coefficients $\tau_{i_k}(u_i)$). So we have, in particular, that the solution $x_1 = a_1, x_2 = a_2, \ldots, x_r = a_r$ and $x_{r+1} = x_{r+2} = \cdots x_{n+1} = 0$ of system (2) is also a solution of system (1). We can now pick up at step $(*)$ and complete the proof by contradiction. □

Lemma V.2.9 (continued 6)

Proof (continued).
(ii) If ζ and θ are both elements in the same coset of H in J, then (since $u_i \in H'$) $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots n+1$.

Sub-Proof. Let $\zeta, \theta \in aH$. Then $\zeta = ah_1$ and $\theta = ah_2$ for some $h_1, h_2 \in H$. Since H' is the fixed field of H and each $u_i \in H'$, then $\zeta(u_i) = (ah_1)(u_i) = ah_1(u_i) = au_i$ and $\theta(u_i) = (ah_2)(u_i) = ah_2(u_i) = au_i$. So $\zeta(u_i) = \theta(u_i)$ for $i = 1, 2, \ldots, n+1$. *Sub-Q.E.D.*

It now follows from claim (i) that there is some reordering $i_1, i_2, \ldots, i_{n+1}$ of $1, 2, \ldots, n+1$ so that for each $k = 1, 2, \ldots, n+1$, $\sigma\tau_k$ and τ_{i_k} are in the same coset of H in J. By (ii), the kth equation of system (2) (with coefficients $\sigma\tau_k(u_i)$) is identical with the i_kth equation of system (1) (with coefficients $\tau_{i_k}(u_i)$). So we have, in particular, that the solution $x_1 = a_1, x_2 = a_2, \ldots, x_r = a_r$ and $x_{r+1} = x_{r+2} = \cdots x_{n+1} = 0$ of system (2) is also a solution of system (1). We can now pick up at step $(*)$ and complete the proof by contradiction. □

Lemma V.2.10

Lemma V.2.10. Let F be an extension field of K, L and M intermediate fields with $L \subset M$, and H, J subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$.

(i) If L is closed and $[M : L]$ finite, then M is closed and $[L' : M'] = [M : L]$;

(ii) if H is closed and $[J : H]$ finite, then J is closed and $[H' : J'] = [J : H]$;

(iii) if F is a finite dimensional Galois extension of K, then all intermediate fields and and all subgroups of the Galois group are closed and $\mathrm{Aut}_K(F)$ has order $[F : K]$.

Proof. **(i)** By Lemma V.2.6(iii), $M \subset M''$. Since $L \subset M \subset M''$, by Theorem V.1.2 we have $[M'' : L] = [M'' : M][M : L]$ and so $[M : L] \leq [M'' : L]$. Now $[L' : M'] \leq [M : L]$ By Lemma V.2.8 and $[M'' : L''] \leq [L' : M']$ by Lemma V.2.9.

Lemma V.2.10

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Lemma V.2.10 (continued 1)

Lemma V.2.10. Let F be an extension field of K, L and M intermediate fields with $L \subset M$, and H, J subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$.

(i) If L is closed and $[M : L]$ finite, then M is closed and $[L' : M'] = [M : L]$.

Proof (continued). (i) Combining these inequalities gives

$$\begin{aligned}[M : L] \leq [M'' : L] &= [M'' : L''] \text{ since } L'' = L \\ &\leq [L' : M'] \leq [M : L].\end{aligned}$$

Therefore the inequalities reduce to equalities and $[L' : M'] = [M : L]$. Also, $[M'' : L] = [M : L]$ so the dimension of M'' over L is the same as the dimension of M over L. Also, by Lemma V.2.6(iii), $M \subset M''$ and so $M = M''$ and M is closed.

Lemma V.2.10 (continued 1)

Lemma V.2.10. Let F be an extension field of K, L and M intermediate fields with $L \subset M$, and H, J subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$.

(i) If L is closed and $[M : L]$ finite, then M is closed and $[L' : M'] = [M : L]$.

Proof (continued). (i) Combining these inequalities gives

$$\begin{aligned} [M : L] \leq [M'' : L] &= [M'' : L''] \text{ since } L'' = L \\ &\leq [L' : M'] \leq [M : L]. \end{aligned}$$

Therefore the inequalities reduce to equalities and $[L' : M'] = [M : L]$. Also, $[M'' : L] = [M : L]$ so the dimension of M'' over L is the same as the dimension of M over L. Also, by Lemma V.2.6(iii), $M \subset M''$ and so $M = M''$ and M is closed.

Lemma V.2.10 (continued 2)

Lemma V.2.10. Let F be an extension field of K, L and M intermediate fields with $L \subset M$, and H, J subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$.

(ii) if H is closed and $[J : H]$ finite, then J is closed and $[H' : J'] = [J : H]$.

Proof. (ii) By Lemma V.2.6(iii), $J < J''$. Since $H < J < J''$ then the number of cosets of H in J, $[J : H]$ is less than or equal to the number of cosets of H in J'', $[J'' : H]$; that is, $[J : H] \leq [J'' : H]$.

Lemma V.2.10 (continued 2)

Lemma V.2.10. Let F be an extension field of K, L and M intermediate fields with $L \subset M$, and H, J subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$.

(ii) if H is closed and $[J : H]$ finite, then J is closed and $[H' : J'] = [J : H]$.

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$$\begin{aligned} [J : H] &\le [J'' : H] = [J'' : H''] \text{ since } H = H'' \\ &\le [H' : J'] \text{ by Lemma V.2.8} \\ &\le [J : H] \text{ by Lemma V.2.9.} \end{aligned}$$

So we have $[H' : J'] = [J : H]$ as claimed.

Lemma V.2.10 (continued 2)

Lemma V.2.10. Let F be an extension field of K, L and M intermediate fields with $L \subset M$, and H, J subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$.

(ii) if H is closed and $[J : H]$ finite, then J is closed and $[H' : J'] = [J : H]$.

Proof. (ii) By Lemma V.2.6(iii), $J < J''$. Since $H < J < J''$ then the number of cosets of H in J, $[J : H]$ is less than or equal to the number of cosets of H in J'', $[J'' : H]$; that is, $[J : H] \le [J'' : H]$. So

$$\begin{aligned} [J : H] &\le [J'' : H] = [J'' : H''] \text{ since } H = H'' \\ &\le [H' : J'] \text{ by Lemma V.2.8} \\ &\le [J : H] \text{ by Lemma V.2.9.} \end{aligned}$$

So we have $[H' : J'] = [J : H]$ as claimed. Also, $[J'' : H] = [J : H]$ and so the number of cosets of H in J equals the number of cosets of H in J''. Therefore $|J| = |J''|$; also $J \subset J''$ so we must have $J = J''$.

Lemma V.2.10 (continued 2)

Lemma V.2.10. Let F be an extension field of K, L and M intermediate fields with $L \subset M$, and H, J subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$.

(ii) if H is closed and $[J : H]$ finite, then J is closed and $[H' : J'] = [J : H]$.

Proof. (ii) By Lemma V.2.6(iii), $J < J''$. Since $H < J < J''$ then the number of cosets of H in J, $[J : H]$ is less than or equal to the number of cosets of H in J'', $[J'' : H]$; that is, $[J : H] \le [J'' : H]$. So

$$\begin{aligned} [J : H] &\le [J'' : H] = [J'' : H''] \text{ since } H = H'' \\ &\le [H' : J'] \text{ by Lemma V.2.8} \\ &\le [J : H] \text{ by Lemma V.2.9.} \end{aligned}$$

So we have $[H' : J'] = [J : H]$ as claimed. Also, $[J'' : H] = [J : H]$ and so the number of cosets of H in J equals the number of cosets of H in J''. Therefore $|J| = |J''|$; also $J \subset J''$ so we must have $J = J''$.

Lemma V.2.10 (continued 3)

Lemma V.2.10. Let F be an extension field of K, L and M intermediate fields with $L \subset M$, and H, J subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$.

(iii) if F is a finite dimensional Galois extension of K, then all intermediate fields and and all subgroups of the Galois group are closed and $\mathrm{Aut}_K(F)$ has order $[F : K]$.

Proof. **(iii)** If E is an intermediate field, $K \subset E \subset F$, then $[F : K] = [F : E][E : K]$ by Theorem V.1.2 and since $[F : K]$ is hypothesized to be finite, then $[E : K]$ is finite.

Lemma V.2.10 (continued 3)

Lemma V.2.10. Let F be an extension field of K, L and M intermediate fields with $L \subset M$, and H, J subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$.

(iii) if F is a finite dimensional Galois extension of K, then all intermediate fields and and all subgroups of the Galois group are closed and $\mathrm{Aut}_K(F)$ has order $[F : K]$.

Proof. (iii) If E is an intermediate field, $K \subset E \subset F$, then $[F : K] = [F : E][E : K]$ by Theorem V.1.2 and since $[F : K]$ is hypothesized to be finite, then $[E : K]$ is finite. Since F is Galois over K then K is closed (see the note on page 246 right after the definition of closed). So every intermediate field is closed.

Lemma V.2.10 (continued 3)

Lemma V.2.10. Let F be an extension field of K, L and M intermediate fields with $L \subset M$, and H, J subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$.

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Proof. (iii) If E is an intermediate field, $K \subset E \subset F$, then $[F : K] = [F : E][E : K]$ by Theorem V.1.2 and since $[F : K]$ is hypothesized to be finite, then $[E : K]$ is finite. Since F is Galois over K then K is closed (see the note on page 246 right after the definition of closed). So every intermediate field is closed. Now (i) (with $L = K$ and $M = E$) implies that E is closed and $[K' : E'] = [E : K]$. In particular, if $E = F$ then $|\mathrm{Aut}_K(F)| = [\mathrm{Aut}_K(E) : 1] = [K' : F'] = [F : K]$ is finite. Therefore, every subgroup J of $\mathrm{Aut}_K(F)$ is finite.

Lemma V.2.10 (continued 3)

Lemma V.2.10. Let F be an extension field of K, L and M intermediate fields with $L \subset M$, and H, J subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$.

(iii) if F is a finite dimensional Galois extension of K, then all intermediate fields and and all subgroups of the Galois group are closed and $\mathrm{Aut}_K(F)$ has order $[F : K]$.

Proof. (iii) If E is an intermediate field, $K \subset E \subset F$, then $[F : K] = [F : E][E : K]$ by Theorem V.1.2 and since $[F : K]$ is hypothesized to be finite, then $[E : K]$ is finite. Since F is Galois over K then K is closed (see the note on page 246 right after the definition of closed). So every intermediate field is closed. Now (i) (with $L = K$ and $M = E$) implies that E is closed and $[K' : E'] = [E : K]$. In particular, if $E = F$ then $|\mathrm{Aut}_K(F)| = [\mathrm{Aut}_K(E) : 1] = [K' : F'] = [F : K]$ is finite. Therefore, every subgroup J of $\mathrm{Aut}_K(F)$ is finite. Now $1' = F$ and $1'' = F' = \mathrm{Aut}_F(F) = 1$, so 1 is closed. Now by (ii), J is closed and so every subgroup of $\mathrm{Aut}_K(F)$ is closed. □

Lemma V.2.10 (continued 3)

Lemma V.2.10. Let F be an extension field of K, L and M intermediate fields with $L \subset M$, and H, J subgroups of the Galois group $\mathrm{Aut}_K(F)$ with $H < J$.

(iii) if F is a finite dimensional Galois extension of K, then all intermediate fields and and all subgroups of the Galois group are closed and $\mathrm{Aut}_K(F)$ has order $[F : K]$.

Proof. **(iii)** If E is an intermediate field, $K \subset E \subset F$, then $[F : K] = [F : E][E : K]$ by Theorem V.1.2 and since $[F : K]$ is hypothesized to be finite, then $[E : K]$ is finite. Since F is Galois over K then K is closed (see the note on page 246 right after the definition of closed). So every intermediate field is closed. Now (i) (with $L = K$ and $M = E$) implies that E is closed and $[K' : E'] = [E : K]$. In particular, if $E = F$ then $|\mathrm{Aut}_K(F)| = [\mathrm{Aut}_K(E) : 1] = [K' : F'] = [F : K]$ is finite. Therefore, every subgroup J of $\mathrm{Aut}_K(F)$ is finite. Now $1' = F$ and $1'' = F' = \mathrm{Aut}_F(F) = 1$, so 1 is closed. Now by (ii), J is closed and so every subgroup of $\mathrm{Aut}_K(F)$ is closed. □

Lemma V.2.11

Lemma V.2.11. Let F be an extension field of K.

(i) If E is a stable intermediate field of the extension, then $E' = \mathrm{Aut}_E(F)$ is a normal subgroup of the Galois group $\mathrm{Aut}_K(F)$;

(ii) if H is a normal subgroup of $\mathrm{Aut}_K(F)$, then the fixed field H' of H is a stable intermediate field of the extension.

Proof. **(i)** If $u \in E$ and $\sigma \in \mathrm{Aut}_K(F)$ then $\sigma(u) \in E$ by the stability of E. Hence for $\tau \in E' = \mathrm{Aut}_E(F)$ we have $\tau\sigma(u) = \sigma(u)$.

Lemma V.2.11

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(i) If E is a stable intermediate field of the extension, then $E' = \mathrm{Aut}_E(F)$ is a normal subgroup of the Galois group $\mathrm{Aut}_K(F)$;

(ii) if H is a normal subgroup of $\mathrm{Aut}_K(F)$, then the fixed field H' of H is a stable intermediate field of the extension.

Proof. (i) If $u \in E$ and $\sigma \in \mathrm{Aut}_K(F)$ then $\sigma(u) \in E$ by the stability of E. Hence for $\tau \in E' = \mathrm{Aut}_E(F)$ we have $\tau\sigma(u) = \sigma(u)$. Therefore, for any $\sigma \in \mathrm{Aut}_K(F)$, $\tau \in E' = \mathrm{Aut}_E(F)$, and $u \in E$ we have $\sigma^{-1}\tau\sigma(u) = \sigma^{-1}\sigma(u) = u$. Consequently $\sigma^{-1}\tau\sigma \in E' = \mathrm{Aut}_E(F)$ and hence E' is a normal subgroup of $\mathrm{Aut}_K(F)$ by Theorem I.5.1(iv).

Lemma V.2.11

Lemma V.2.11. Let F be an extension field of K.

(i) If E is a stable intermediate field of the extension, then $E' = \text{Aut}_E(F)$ is a normal subgroup of the Galois group $\text{Aut}_K(F)$;

(ii) if H is a normal subgroup of $\text{Aut}_K(F)$, then the fixed field H' of H is a stable intermediate field of the extension.

Proof. (i) If $u \in E$ and $\sigma \in \text{Aut}_K(F)$ then $\sigma(u) \in E$ by the stability of E. Hence for $\tau \in E' = \text{Aut}_E(F)$ we have $\tau\sigma(u) = \sigma(u)$. Therefore, for any $\sigma \in \text{Aut}_K(F)$, $\tau \in E' = \text{Aut}_E(F)$, and $u \in E$ we have $\sigma^{-1}\tau\sigma(u) = \sigma^{-1}\sigma(u) = u$. Consequently $\sigma^{-1}\tau\sigma \in E' = \text{Aut}_E(F)$ and hence E' is a normal subgroup of $\text{Aut}_K(F)$ by Theorem I.5.1(iv).

Lemma V.2.11 (continued)

Lemma V.2.11. Let F be an extension field of K.

(ii) if H is a normal subgroup of $\text{Aut}_K(F)$, then the fixed field H' of H is a stable intermediate field of the extension.

Proof. (ii) If $\sigma \in \text{Aut}_K(F)$ and $\tau \in H$, then $\sigma^{-1}\tau\sigma \in H$ since H is hypothesized to be a normal subgroup of $\text{Aut}_K(F)$ (by Theorem I.5.1(iv)). Therefore, for any $u \in H'$, $\sigma^{-1}\tau\sigma(u) = u$ (since H' denotes the fixed field of H), which implies that $\tau\sigma(u) = \sigma(u)$ for all $\tau \in H$.

Lemma V.2.11 (continued)

Lemma V.2.11. Let F be an extension field of K.

(ii) if H is a normal subgroup of $\mathrm{Aut}_K(F)$, then the fixed field H' of H is a stable intermediate field of the extension.

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Lemma V.2.11 (continued)

Lemma V.2.11. Let F be an extension field of K.

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Lemma V.2.12

Lemma V.2.12. If F is a Galois extension field of K and E is a stable intermediate field of the extension, then E is Galois over K.

Proof. If $u \in E \setminus K$ then there exists $\sigma \in \text{Aut}_K(F)$ such that $\sigma(u) \neq u$ since F is Galois over K (meaning $K = (\text{Aut}_K(F))' = K'$, so $u \notin K$ implies that u is not fixed by some $\sigma \in \text{Aut}_K(F)$).

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Proof. If $u \in E \setminus K$ then there exists $\sigma \in \text{Aut}_K(F)$ such that $\sigma(u) \neq u$ since F is Galois over K (meaning $K = (\text{Aut}_K(F))' = K'$, so $u \notin K$ implies that u is not fixed by some $\sigma \in \text{Aut}_K(F)$). Since E is stable then σ maps E into itself; that is, $\sigma|_E \in \text{Aut}_K(E)$. So for every $u \in E \setminus K$ there is an element of $\text{Aut}_K(F)$ which does not fix u. So the fixed field of $\text{Aut}_K(F)$ is just K; $K = (\text{Aut}_K(F))' = K'$. Therefore, E is a Galois extension of K. □

Lemma V.2.13

Lemma V.2.13. If F is an extension field of K and E is an intermediate field of the extension such that E is algebraic and Galois over K, then E is stable (relative to F and K).

Proof. If $u \in E$, let $f \in K[x]$ be the irreducible monic polynomial of u and let $u_1, u_2, \ldots, u_r$ be the distinct roots of f that lie in E, where $u = u_1$. Then $r \leq n = \deg(f)$ by Theorem III.6.7.

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Lemma V.2.13 (continued)

Lemma V.2.13. If F is an extension field of K and E is an intermediate field of the extension such that E is algebraic and Galois over K, then E is stable (relative to F and K).

Proof (continued). Now if $\sigma \in \text{Aut}_K(F)$, then $\sigma(u)$ is a root of f by Theorem V.2.2, whence $\sigma(u) \in E$. Since u was an arbitrary element of E, we have shown that $\sigma \in \text{Aut}_K(F)$ maps E into itself; that is, E is stable relative to K and F. □

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Lemma V.2.14

Lemma V.2.14. Let F be an extension field of K and let E be a stable intermediate field of the extension. Then the quotient group $\mathrm{Aut}_K(F)/\mathrm{Aut}_E(F)$ is isomorphic to the group of all automorphisms in $\mathrm{Aut}_K(E)$ that are extendible to F.

Proof. Intermediate field E is stable, so (by the definition of stable) every automorphism $\sigma \in \mathrm{Aut}_K(F)$ maps E into itself, and hence the mapping $\sigma \mapsto \sigma|_E$ defines a group homomorphism from $\mathrm{Aut}_K(F)$ to $\mathrm{Aut}_K(E)$. The image of this homomorphism is "clearly" the subgroup of $\mathrm{Aut}_K(E)$ of all automorphisms that are extendible to F (of course, the extension of $\sigma|_E$ is σ itself).

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Theorem V.2.5. The Fundamental Theorem of Galois Theory

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If F is a finite dimensional Galois extension of K, then there is a one to one correspondence between the set of all intermediate fields of the extension and the set of all subgroups of the Galois group $\mathrm{Aut}_K(F)$ (given by $E \mapsto E' = \mathrm{Aut}_E(F)$) such that:

(i) the relative dimension of two intermediate fields is equal to the relative index of the corresponding subgroups; in particular, $\mathrm{Aut}_K(F)$ has order $[F : K]$;

(ii) F is Galois over every intermediate field E, but E is Galois over K if and only if the corresponding subgroup $E' = \mathrm{Aut}_E(F)$ is normal in $G = \mathrm{Aut}_K(F)$; in the case G/E' is (isomorphic to) the Galois group $\mathrm{Aut}_K(E)$ of E over K.

Theorem V.2.5. The Fundamental Theorem of Galois Theory (i)

Proof. Theorem V.2.7 shows that there is a one to one correspondence between the *closed* intermediate fields and *closed* subgroups of the Galois group. By Lemma V.2.10(iii) all intermediate fields are closed and all subgroups of $\text{Aut}_K(F)$ are closed. So the one to one correspondence between closed intermediate fields and closed subgroups is in fact a one to one correspondence between all intermediate fields and all subgroups. This correspondence is given by mapping each group H to its fixed field H' and by mapping each field M to its Galois group $M' = \text{Aut}_M(F)$.

Theorem V.2.5. The Fundamental Theorem of Galois Theory (i)

Proof. Theorem V.2.7 shows that there is a one to one correspondence between the *closed* intermediate fields and *closed* subgroups of the Galois group. By Lemma V.2.10(iii) all intermediate fields are closed and all subgroups of $\text{Aut}_K(F)$ are closed. So the one to one correspondence between closed intermediate fields and closed subgroups is in fact a one to one correspondence between all intermediate fields and all subgroups. This correspondence is given by mapping each group H to its fixed field H' and by mapping each field M to its Galois group $M' = \text{Aut}_M(F)$.

(i) For intermediate fields L and M (with $L \subset M$) we have by Lemma V.2.10(i) that the relative dimension of the fields $[M : L]$ equals the relative index of the corresponding subgroups $[L' : M']$; that is, $[M : L] = [L' : M']$.

Theorem V.2.5. The Fundamental Theorem of Galois Theory (i)

Proof. Theorem V.2.7 shows that there is a one to one correspondence between the *closed* intermediate fields and *closed* subgroups of the Galois group. By Lemma V.2.10(iii) all intermediate fields are closed and all subgroups of $\mathrm{Aut}_K(F)$ are closed. So the one to one correspondence between closed intermediate fields and closed subgroups is in fact a one to one correspondence between all intermediate fields and all subgroups. This correspondence is given by mapping each group H to its fixed field H' and by mapping each field M to its Galois group $M' = \mathrm{Aut}_M(F)$.

(i) For intermediate fields L and M (with $L \subset M$) we have by Lemma V.2.10(i) that the relative dimension of the fields $[M : L]$ equals the relative index of the corresponding subgroups $[L' : M']$; that is, $[M : L] = [L' : M']$. The "in particular" part follows from Lemma V.2.10(iii); that is, $|\mathrm{Aut}_K(F)| = [F : K]$.

Theorem V.2.5. The Fundamental Theorem of Galois Theory (i)

Proof. Theorem V.2.7 shows that there is a one to one correspondence between the *closed* intermediate fields and *closed* subgroups of the Galois group. By Lemma V.2.10(iii) all intermediate fields are closed and all subgroups of $\text{Aut}_K(F)$ are closed. So the one to one correspondence between closed intermediate fields and closed subgroups is in fact a one to one correspondence between all intermediate fields and all subgroups. This correspondence is given by mapping each group H to its fixed field H' and by mapping each field M to its Galois group $M' = \text{Aut}_M(F)$.

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Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii)

(ii) F is Galois over every intermediate field E, but E is Galois over K if and only if the corresponding subgroup $E' = \text{Aut}_E(F)$ is normal in $G = \text{Aut}_K(F)$; in the case G/E' is (isomorphic to) the Galois group $\text{Aut}_K(E)$ of E over K.

Proof. F is Galois over E since E is closed (see the comment after the definition of closed), so F is Galois over every intermediate field. E is finite dimensional over K (since F is; see Theorem V.1.2) and hence, by Theorem V.1.11, F is algebraic over K. Consequently if E is Galois over K then E satisfies the hypotheses of Lemma V.2.13 and so E is stable relative to F and K. By Lemma V.2.11(i), $E' = \text{Aut}_E(F)$ is normal in $G = \text{Aut}_K(F)$ (this is the first part of the claim of (ii)).

Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii)

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Proof. F is Galois over E since E is closed (see the comment after the definition of closed), so F is Galois over every intermediate field. E is finite dimensional over K (since F is; see Theorem V.1.2) and hence, by Theorem V.1.11, F is algebraic over K. Consequently if E is Galois over K then E satisfies the hypotheses of Lemma V.2.13 and so E is stable relative to F and K. By Lemma V.2.11(i), $E' = \mathrm{Aut}_E(F)$ is normal in $G = \mathrm{Aut}_K(F)$ (this is the first part of the claim of (ii)). Conversely, if $E' = \mathrm{Aut}_E(F)$ is normal in $G = \mathrm{Aut}_K(F)$, then by Lemma V.2.11(ii), E'' is a stable intermediate field.

Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii)

(ii) F is Galois over every intermediate field E, but E is Galois over K if and only if the corresponding subgroup $E' = \text{Aut}_E(F)$ is normal in $G = \text{Aut}_K(F)$; in the case G/E' is (isomorphic to) the Galois group $\text{Aut}_K(E)$ of E over K.

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Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii) (continued)

Proof (continued). We establish that all intermediate fields are closed, so $E = E''$ and E is stable. Therefore by Lemma V.2.12, E is Galois over K (this is the second part of the claim of (ii), the converse of the first part).

Now for the "in the case" part.

Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii) (continued)

Proof (continued). We establish that all intermediate fields are closed, so $E = E''$ and E is stable. Therefore by Lemma V.2.12, E is Galois over K (this is the second part of the claim of (ii), the converse of the first part).

Now for the "in the case" part. Let $E' = \text{Aut}_E(F)$ be normal in $G = \text{Aut}_K(F)$, or equivalently, let E be Galois over K. We have seen at the beginning of the proof that all intermediate fields and subgroups are closed, so E and E' are closed. Since F is Galois over K then $G' = (\text{Aut}_K(F))' = K$. Now the elements of G/E' are cosets of E', so $|G/E'| = [G : E']$.

Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii) (continued)

Proof (continued). We establish that all intermediate fields are closed, so $E = E''$ and E is stable. Therefore by Lemma V.2.12, E is Galois over K (this is the second part of the claim of (ii), the converse of the first part).

Now for the "in the case" part. Let $E' = \text{Aut}_E(F)$ be normal in $G = \text{Aut}_K(F)$, or equivalently, let E be Galois over K. We have seen at the beginning of the proof that all intermediate fields and subgroups are closed, so E and E' are closed. Since F is Galois over K then $G' = (\text{Aut}_K(F))' = K$. Now the elements of G/E' are cosets of E', so $|G/E'| = [G : E']$. Hence

$$\begin{aligned} |G/E'| &= [G : E'] \\ &= [E'' : G'] \text{ by Lemma V.2.10(ii)} \\ &= [E : K] \text{ since } E = E'' \text{ and } K = G'. \end{aligned}$$

Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii) (continued)

Proof (continued). We establish that all intermediate fields are closed, so $E = E''$ and E is stable. Therefore by Lemma V.2.12, E is Galois over K (this is the second part of the claim of (ii), the converse of the first part).

Now for the "in the case" part. Let $E' = \text{Aut}_E(F)$ be normal in $G = \text{Aut}_K(F)$, or equivalently, let E be Galois over K. We have seen at the beginning of the proof that all intermediate fields and subgroups are closed, so E and E' are closed. Since F is Galois over K then $G' = (\text{Aut}_K(F))' = K$. Now the elements of G/E' are cosets of E', so $|G/E'| = [G : E']$. Hence

$$\begin{aligned} |G/E'| &= [G : E'] \\ &= [E'' : G'] \text{ by Lemma V.2.10(ii)} \\ &= [E : K] \text{ since } E = E'' \text{ and } K = G'. \end{aligned}$$

Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii)

(ii) F is Galois over every intermediate field E, but E is Galois over K if and only if the corresponding subgroup $E' = \mathrm{Aut}_E(F)$ is normal in $G = \mathrm{Aut}_K(F)$; in the case G/E' is (isomorphic to) the Galois group $\mathrm{Aut}_K(E)$ of E over K.

Proof. We saw above that E'' is stable and $E = E''$, so E is stable. By Lemma V.2.14, $G/E' = \mathrm{Aut}_K(F)/\mathrm{Aut}_E(F)$ is isomorphic to a subgroup of $\mathrm{Aut}_K(E)$. Since we have just shown that $|G/E'| = [E : K]$, then this subgroup of $\mathrm{Aut}_K(E)$ is of order $[E : K]$. Since E is Galois over K (by hypothesis, here) then part (i) shows that $|\mathrm{Aut}_K(E)| = [E : K]$.

Theorem V.2.5. The Fundamental Theorem of Galois Theory (ii)

(ii) F is Galois over every intermediate field E, but E is Galois over K if and only if the corresponding subgroup $E' = \mathrm{Aut}_E(F)$ is normal in $G = \mathrm{Aut}_K(F)$; in the case G/E' is (isomorphic to) the Galois group $\mathrm{Aut}_K(E)$ of E over K.

Proof. We saw above that E'' is stable and $E = E''$, so E is stable. By Lemma V.2.14, $G/E' = \mathrm{Aut}_K(F)/\mathrm{Aut}_E(F)$ is isomorphic to a subgroup of $\mathrm{Aut}_K(E)$. Since we have just shown that $|G/E'| = [E : K]$, then this subgroup of $\mathrm{Aut}_K(E)$ is of order $[E : K]$. Since E is Galois over K (by hypothesis, here) then part (i) shows that $|\mathrm{Aut}_K(E)| = [E : K]$. Since G/E' is isomorphic to a subgroup of $\mathrm{Aut}_K(E)$ of order $[E : K]$ and $\mathrm{Aut}_K(E)$ itself is of order $[E : K]$, then $G/E' = \mathrm{Aut}_K(F)/\mathrm{Aut}_E(F) \cong \mathrm{Aut}_K(E)$. □

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Theorem V.2.15. Artin

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Let F be a field, G a group of automorphisms of F, and K the fixed field of G in F. Then F is Galois over K. If G is finite, then F is a finite dimensional Galois extension of K with Galois group G.

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Proof (continued). If G is finite, then by Lemma V.2.9 (with $H = 1$ and $J = G$, which gives $|G| = [G : 1]$ is finite) we have

$$\begin{aligned}[F : K] &= [1' : G'] \text{ since } 1' = F \text{ and } G' = K \\ &\qquad \text{(fixed fields of 1 and } G\text{, respectively)} \\ &\leq [G : 1] \text{ by Lemma V.2.9} \\ &= |G|.\end{aligned}$$

Consequently, F is finite dimensional over K. So F is a finite dimensional Galois extension of K, and so by Lemma V.2.10(iii) all intermediate groups are closed and so $G = G''$. Since the fixed field of G is $G' = K$ (and hence $G'' = K'$) we have that the Galois group of F over K is

$$\begin{aligned}\mathrm{Aut}_K(F) &= K' \text{ by the prime notation} \\ &= G'' \text{ as just observed} \\ &= G \text{ since } G \text{ is closed.}\end{aligned}$$

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