Theorem V.6.4. If $F$ is an algebraic extension field of a field $K$ of characteristic $p \neq 0$, then the following statements are equivalent:

1. $F$ is an algebraic extension field of a field $K$ of characteristic $p \neq 0$.
2. $F$ is separable over $K$.
3. $F$ is purely inseparable over $K$.

Proof. Let $F$ be an extension field of $K$. If $F$ is both separable and purely inseparable over $K$, then $F$ is the field $K(x) = K[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ are algebraically independent over $K$. If $F$ is both separable and purely inseparable over $K$, then $F$ is the field $K(x) = K[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ are algebraically independent over $K$.

Lemma V.6.3. Let $F$ be a field, $K$ a subfield of $F$, and $a \in F$. Then $a$ is separable over $K$ if and only if $a$ occurs in characteristic $p \neq 0$.

Proof. If $a$ is separable over $K$, then $a$ occurs in characteristic $p \neq 0$. Conversely, if $a$ occurs in characteristic $p \neq 0$, then $a$ is separable over $K$. 

\[ a \in K \iff \text{char}(K) = p \neq 0 \]

Exercise V.6.1. Let $K$ be a field of characteristic $p \neq 0$, and let $n \in \mathbb{N}$. Then $a$ is separable over $K$ if and only if $a$ occurs in characteristic $p \neq 0$.

Proof. If $a$ is separable over $K$, then $a$ occurs in characteristic $p \neq 0$. Conversely, if $a$ occurs in characteristic $p \neq 0$, then $a$ is separable over $K$. 

Theorem V.6.2. Let $F$ be an algebraic extension field of $K$. Then $F$ is separable over $K$ if and only if $F$ is purely inseparable over $K$. 

Proof. If $F$ is separable over $K$, then $F$ is purely inseparable over $K$. Conversely, if $F$ is purely inseparable over $K$, then $F$ is separable over $K$. 

Exercise V.6.3. Let $F$ be an algebraic extension field of $K$. Then $F$ is separable over $K$ if and only if $F$ is purely inseparable over $K$. 

Proof. If $F$ is separable over $K$, then $F$ is purely inseparable over $K$. Conversely, if $F$ is purely inseparable over $K$, then $F$ is separable over $K$. 

\[ a \in K \iff \text{char}(K) = p \neq 0 \]
are elements of $K$ itself, and (iv) holds.

**Theorem V.6.4.:** If $K$ is an algebraic extension field of a field $F$ of characteristic different from 0, then the following statements are equivalent:

1. $F$ is a separable extension field of $K$.
2. For every irreducible polynomial $f \in F[x]$ with $f(0) \neq 0$, there exists an integer $n \neq 0$ such that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$.
3. For every irreducible polynomial $f \in F[x]$ with $f(0) \neq 0$, there exists an integer $n \neq 0$ such that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and $\nu(f) > 0$.

**Proof:** (i) $\Rightarrow$ (ii) Let $f \in F[x]$, where $f(0) \neq 0$. Then $f(x)$ is separable and irreducible over $K$. Thus, $f(0) \neq 0$ and $\nu(f) > 0$.

(ii) $\Rightarrow$ (iii) Let $f \in F[x]$, where $f(0) \neq 0$. Then $f(x)$ is separable and irreducible over $K$. Thus, $f(0) \neq 0$ and $\nu(f) > 0$.

(iii) $\Rightarrow$ (i) Let $f \in F[x]$, where $f(0) \neq 0$. Then $f(x)$ is separable and irreducible over $K$. Thus, $f(0) \neq 0$ and $\nu(f) > 0$.

**Theorem V.6.5.:** If $K$ is an algebraic extension field of a field $F$ of characteristic different from 0, then the following statements are equivalent:

1. $F$ is a separable extension field of $K$.
2. For every irreducible polynomial $f \in F[x]$ with $f(0) \neq 0$, there exists an integer $n \neq 0$ such that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and $\nu(f) > 0$.
3. For every irreducible polynomial $f \in F[x]$ with $f(0) \neq 0$, there exists an integer $n \neq 0$ such that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and $\nu(f) > 0$.

**Proof:** (i) $\Rightarrow$ (ii) Let $f \in F[x]$, where $f(0) \neq 0$. Then $f(x)$ is separable and irreducible over $K$. Thus, $f(0) \neq 0$ and $\nu(f) > 0$.

(ii) $\Rightarrow$ (iii) Let $f \in F[x]$, where $f(0) \neq 0$. Then $f(x)$ is separable and irreducible over $K$. Thus, $f(0) \neq 0$ and $\nu(f) > 0$.

(iii) $\Rightarrow$ (i) Let $f \in F[x]$, where $f(0) \neq 0$. Then $f(x)$ is separable and irreducible over $K$. Thus, $f(0) \neq 0$ and $\nu(f) > 0$.
Corollary V.6.5: If \( K \) is a finite-dimensional purely inseparable extension of \( \mathbb{F} \), then \( [K: \mathbb{F}] \neq 0 \) if and only if \( \text{char}(K) = p \). In this case, \( K \) is generated by \( n \) elements over \( \mathbb{F} \), where \( n \equiv 0 \) (mod \( p \)).

Theorem V.6.7 (continued): Suppose the only elements of \( F \) which are separable over \( \mathbb{F} \) are the characteristic \( p \) elements of \( F \), then the following statements are equivalent:

1. \( F \) is an algebraic extension field of a field \( K \) of characteristic \( p \).
2. \( F \) is an algebraic extension field of a field \( A \) of characteristic \( p \).
3. For some \( n \geq 0 \), \( F = \langle \alpha_1, \ldots, \alpha_n \rangle \) for \( \alpha_1, \ldots, \alpha_n \in A \) and \( \alpha_i \neq 0 \) for some \( i \).
4. \( F = \langle \alpha_1, \ldots, \alpha_n \rangle \) for \( \alpha_1, \ldots, \alpha_n \in A \) and \( \alpha_i \neq 0 \) for some \( i \).
5. \( F = \langle \alpha_1, \ldots, \alpha_n \rangle \) for \( \alpha_1, \ldots, \alpha_n \in A \) and \( \alpha_i \neq 0 \) for some \( i \).
6. \( F = \langle \alpha_1, \ldots, \alpha_n \rangle \) for \( \alpha_1, \ldots, \alpha_n \in A \) and \( \alpha_i \neq 0 \) for some \( i \).
7. \( F = \langle \alpha_1, \ldots, \alpha_n \rangle \) for \( \alpha_1, \ldots, \alpha_n \in A \) and \( \alpha_i \neq 0 \) for some \( i \).
8. \( F = \langle \alpha_1, \ldots, \alpha_n \rangle \) for \( \alpha_1, \ldots, \alpha_n \in A \) and \( \alpha_i \neq 0 \) for some \( i \).
9. \( F = \langle \alpha_1, \ldots, \alpha_n \rangle \) for \( \alpha_1, \ldots, \alpha_n \in A \) and \( \alpha_i \neq 0 \) for some \( i \).
10. \( F = \langle \alpha_1, \ldots, \alpha_n \rangle \) for \( \alpha_1, \ldots, \alpha_n \in A \) and \( \alpha_i \neq 0 \) for some \( i \).

Proof: (v) Suppose the only elements of \( F \) which are separable over \( \mathbb{F} \) are the characteristic \( p \) elements of \( F \), then the following statements are equivalent:
So by Theorem \( \{6.3\} \), \( F \cup S = \mathbb{K} \). The elements of \( F \cup S \) are both separable and purely inseparable over \( \mathbb{K} \).

(iii) The set of elements of \( F \cup S \) that are separable and purely inseparable over \( \mathbb{K} \) is separable over \( \mathbb{K} \).

Proof (continued). (iii) By Theorem \( \{6.2\} \), every element \( x \in F \cup S \) is separable or purely inseparable over \( \mathbb{K} \).

\[ \mathbb{K} = F \cup S \]

If \( x \in F \), then \( x \) is purely inseparable over \( \mathbb{K} \).

If \( x \in S \), then \( x \) is separable over \( \mathbb{K} \).

Theorem \( \{6.7\} \). Let \( F \) be an algebraic extension field of \( K \), let \( S \subseteq F \) be the set of all elements of \( F \) which are separable over \( K \), and let \( P \) be the set of all elements of \( F \) which are purely inseparable over \( K \). Then \( F = K(S) \).

Theorem \( \{6.7\} \). Let \( F \) be an algebraic extension field of \( K \), let \( S \subseteq F \) be the set of all elements of \( F \) which are separable over \( K \), and let \( P \) be the set of all elements of \( F \) which are purely inseparable over \( K \). Then \( F = K(S) \).

Proof (continued). (iii) By Theorem \( \{6.3\} \), the set of elements of \( F \cup S \) that are separable and purely inseparable over \( \mathbb{K} \) is separable over \( \mathbb{K} \).

So every element of \( F \cup S \) is separable over \( \mathbb{K} \) if \( F \) is a finite subset of \( F \) such that \( F \subseteq K(X) \), and every element of \( X \) is separable over \( \mathbb{K} \).
Corollary V.6.9 (continued 1)

Corollary V.6.9 (continued 2)

Corollary V.6.8
where \( E \) and \( F \) are.

\( f \) and \( g \) are similar. With \( \mathcal{F} \) as the index set for the \( \mathcal{F} \)-tuple.

The proof for \( \mathcal{F} \) not finite is similar. With \( \mathcal{F} \) as the index set for the \( \mathcal{F} \)-tuple.

\( f \) and \( g \) is in the collection \( \mathcal{M} \) above.

\( \mathcal{M} \) is the complete collection of such maps. So \( \mathcal{M} \) is a \( \kappa \)-monomorphism.

Thus \( \mathcal{M} \) is a \( \kappa \)-monomorphism. Then \( \mathcal{M} \) is also a splitting diagram over \( K \).

By Exercise 8.1, \( N \) is a splitting field over \( K \).

Then \( \mathcal{M} \) is a \( \kappa \)-monomorphism. So \( \mathcal{M} \) is a \( \kappa \)-monomorphism. Since \( \mathcal{M} \) is a splitting diagram over \( K \), \( N \) is an algebraic extension field of \( K \). Then \( \mathcal{M} \) is a \( \kappa \)-monomorphism. Since \( \mathcal{M} \) is a splitting diagram over \( K \), \( N \) is an algebraic extension field of \( K \).

Lemma 7.6.11. Let \( F \) be an algebraic extension field of \( E \). Then \( F \) is an algebraic extension field of \( E \).

\( \kappa \)-monomorphisms of \( F \). We also denote the extension \( K \).

\( \kappa \)-monomorphisms, \( \mathcal{M} \) and \( \mathcal{M} \). For \( \mathcal{F} \) of \( \mathcal{M} \)-3.8 as \( \mathcal{M} \), each \( \mathcal{M} \) extends to a field over \( E \) of the same set of \( \mathcal{M} \)-3.2. Since \( \mathcal{M} \) is a splitting diagram over \( K \), \( N \) is a splitting field over \( K \).

Then \( \mathcal{M} \) is a \( \kappa \)-monomorphism. So \( \mathcal{M} \) is a \( \kappa \)-monomorphism. Since \( \mathcal{M} \) is a splitting diagram over \( K \), \( N \) is an algebraic extension field of \( K \).

Lemma 7.6.11. Let \( F \) be an algebraic extension field of \( E \). Then \( F \) is an algebraic extension field of \( E \).

\( \kappa \)-monomorphisms of \( F \). We also denote the extension \( K \).

\( \kappa \)-monomorphisms, \( \mathcal{M} \) and \( \mathcal{M} \). For \( \mathcal{F} \) of \( \mathcal{M} \)-3.8 as \( \mathcal{M} \), each \( \mathcal{M} \) extends to a field over \( E \) of the same set of \( \mathcal{M} \)-3.2. Since \( \mathcal{M} \) is a splitting diagram over \( K \), \( N \) is a splitting field over \( K \).

Then \( \mathcal{M} \) is a \( \kappa \)-monomorphism. So \( \mathcal{M} \) is a \( \kappa \)-monomorphism. Since \( \mathcal{M} \) is a splitting diagram over \( K \), \( N \) is an algebraic extension field of \( K \).

Lemma 7.6.11. Let \( F \) be an algebraic extension field of \( E \). Then \( F \) is an algebraic extension field of \( E \).

\( \kappa \)-monomorphisms of \( F \). We also denote the extension \( K \).

\( \kappa \)-monomorphisms, \( \mathcal{M} \) and \( \mathcal{M} \). For \( \mathcal{F} \) of \( \mathcal{M} \)-3.8 as \( \mathcal{M} \), each \( \mathcal{M} \) extends to a field over \( E \) of the same set of \( \mathcal{M} \)-3.2. Since \( \mathcal{M} \) is a splitting diagram over \( K \), \( N \) is a splitting field over \( K \).

Then \( \mathcal{M} \) is a \( \kappa \)-monomorphism. So \( \mathcal{M} \) is a \( \kappa \)-monomorphism. Since \( \mathcal{M} \) is a splitting diagram over \( K \), \( N \) is an algebraic extension field of \( K \).

Lemma 7.6.11. Let \( F \) be an algebraic extension field of \( E \). Then \( F \) is an algebraic extension field of \( E \).
Proposition V.6.12 (continued)  

Definition V.1.2  

Remark after Definition V.6.10.  

Proposition V.6.12 (continued)
Proposition V.6.14 (continued)

\[ (j) \seteq [\lambda : (\mathbb{K}) \lambda] \text{ and } n_x = \lambda = \lambda \text{ is the number of distinct roots of } \lambda \text{ in } \mathbb{K} \seteq \{ \lambda : (\mathbb{K}) \lambda \}. \]

After Definition V.6.10, the remark, since every root of a polynomial of degree \( j \) is of multiplicity \( j \) in \( \mathbb{K} \), we must have that \( n_x = \lambda \). Similarly, by Theorem III.6.14 and the number of distinct roots of \( \lambda \) is \( n_x = \lambda \).

Since \( u \) is a splitting field of \( \lambda \) over \( \mathbb{K} \), the proof of Proposition V.6.12 is already done.

Proof (continued).

Corollary V.6.14 (continued)

The result now holds by induction. Corollary V.6.14 follows:\n
\[ f = (\mathbb{K}) \lambda \text{ such that } \lambda \text{ is not an irreducible polynomial in } \mathbb{K}. \]

Theorem V.6.2 shows that if \( f \) is a polynomial with no multiple roots, then \( f \) is separable over \( \mathbb{K} \). Hence, for each \( \lambda \) in \( \mathbb{K} \), \( \lambda \) is separable over \( \mathbb{K} \).

Let \( \lambda \) be an irreducible monic polynomial over \( \mathbb{K} \). Then \( \lambda \) has a splitting field over \( \mathbb{K} \) and \( \lambda \) is separable over \( \mathbb{K} \).

\[ \| \]
Proof (continued). (ii) Now suppose $\mathbf{k}$ is infinite and that $F$ is a finite dimensional extension of $\mathbf{k}.$ Consider the maximal (simple) intermediate field $K,$ i.e., $\mathbf{k} \subset F \subset K,$ and let $\mathbf{v} \in \mathbf{k}.^*$ Since $F$ is an infinite field, there are infinitely many elements $\mathbf{v} \in \mathbf{V}$ with $\mathbf{V} = \text{image}(\mathbf{v}).$ Therefore, there can be only a finite number of intermediate fields $K.$ But if $F$ is any extension field of $\mathbf{k}.$ Let $F$ be a finite dimensional extension field of $\mathbf{k}.$

Proposition V.6.15 (continued II)
Fields.

Conversely, there are only a finite number of intermediate divisors. Consequently, there can have only a finite number of distinct monic splitting fields (by Corollary 11.6.4) \( F[x] \) is a unique factorization domain for any field \( \mathbb{F} \). Thus every \( \mathbb{F} \) is the monic irreducible polynomial of a monic irreducible polynomial of \( \mathbb{F} \).

Intermediates field \( \mathbb{F} \) is uniquely determined by the irreducible monic polynomial. Thus every monic \( \mathbb{F} \) is the monic irreducible polynomial of a monic irreducible polynomial of \( \mathbb{F} \).

Proposition 11.6.4 (continued)