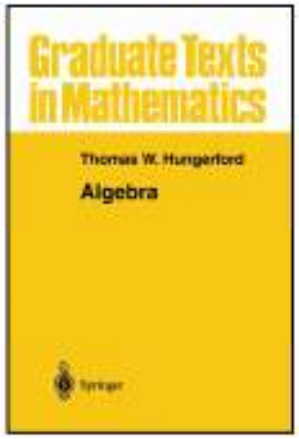
Modern Algebra

Chapter V. Fields and Galois Theory

V.9. Radical Extensions—Proofs of Theorems



Lemma V.9.3

Lemma V.9.3. If F is a radical extension of K and N is a normal closure of F over K (see Theorem V.3.16 on page 265), then N is a radical extension of K.

Proof. The proof is based on two claims.

Claim 1. If F is any finite dimensional extension of K (not necessarily a radical extension) and N is the normal closure of F over K, then N is the composite field $E_1E_2\cdots E_r$ (that is, the subfield of N generated by $E_1 \cup E_2 \cup \cdots \cup E_r$) where each E_i is a subfield of N generated by $E_1 \cup E_2 \cup \cdots \cup E_r$) where each E_i is a subfield of N which is K-isomorphic to F.

Proof 1. Since we hypothesize that F is a finite dimensional extension of K, let $\{w_1, w_2, \ldots, w_n\}$ be a basis of F over K and let f_i be the irreducible polynomial of w_i over K (finite dimensional extensions are algebraic extensions by Theorem V.1.11).

Lemma V.9.3 (continued 1)

Proof (continued). Since N is the normal closure of F over K then as shown in the proof of Theorem V.3.16(i) N is a splitting field of $\{f_1, f_2, \ldots, f_n\}$ over K. For a given f_j, let v be any root of $f_j \in K[x]$ in N. By Theorem V.1.8(ii), since w_i is also a root of $f_j \in K[x]$, then the identity $\iota : K \to K$ extends to an isomorphism $\sigma : K(w_j) \to K(v)$ such that $\sigma(w_j) = v$ (here we let $L = K$ in Theorem V.1.8(ii); that is, σ is a K-isomorphism mapping $K(w_j) \to K(v)$ where $\sigma(w_j) = v$. By Theorem V.3.8 (with $L = K$, $S = \{f_i\}$, $S' = \{\sigma f_i\} = \{f_i\}$, and $F = M = N$) σ extends to a K-automorphism τ of N. Since F is a subfield of N which is isomorphic to F (i.e., $\tau(F) \cong F$) and $w_j \in F$ then $\tau(w_j) = \sigma(w_j) = v \in \tau(F)$. In this way we can find for every root v of every f_i a subfield E of N such that $v \in E$ and E is K-isomorphic to F (the K-isomorphism if τ, as constructed above).

Lemma V.9.3 (continued 2)

Proof (continued). If $E_1, E_2, \ldots, E_r$ are the subfields so obtained, then the subfield of N generated by $E_1 \cup E_2 \cup \cdots \cup E_r$ (that is, the "composite field" $E_1E_2\cdots E_r$) contains all the roots of $f_1, f_2, \ldots, f_n$. That is, $E_1E_2\cdots E_r$ is a splitting field for $\{f_1, f_2, \ldots, f_n\}$ and so by Theorem V.3.14 (the (ii)⇒(i) part) field $E_1E_2\cdots E_r \subset N$ is normal over K. When we have the case $v = w_j$ then the K-isomorphism $\tau : N \to N$ is then identity (since the corresponding $\sigma : K(w_j) \to K(v)$ is the identity) and in this case $\tau(F) = F$ and F is a subfield of the corresponding E_j. So F is a subfield of the composite field $E_1E_2\cdots E_r \subset N$. But by Theorem V.3.16(ii), no proper subfield of N containing F is normal over K, so it must be that $N = E_1E_2\cdots E_r$, proving Claim 1.

Lemma V.9.3 (continued 3)

Proof (continued).
Claim 2. If $E_1, E_2, \ldots, E_r$ are each radical extensions of K, then the composite field $E_1E_2\cdots E_r$ is a radical extension of K.
Proof 2. If E_k is a radical extension of K then (by definition) $E_k = K(u_1^k, u_2^k, \ldots, u_{n_k}^k)$ where some power of u_i^k lies in K and for each $i \geq 2$, some power of u_i^k lies in $K(u_1^k, u_2^k, \ldots, u_{i-1}^k)$. Then $E_1E_2\cdots E_r = K(u_1^1, 2_2^1, \ldots, u_{n_1}^1, u_1^2, u_2^2, \ldots, u_{n_2}^2, u_1^3, u_2^3, \ldots, u_{n_r}^r)$ is "clearly" a radical extension of K, proving Claim 2.
Proof of Lemma. Since by definition, a radical extension is a finite extension, Claim 1 implies that $N = E_1E_2\cdots E_r$ where each E_i is a subfield of N which is K-isomorphic to F. Since F is hypothesized to be a radical extension of K, then each E_i is a radical extension of K. By Claim 2, $N = E_1E_2\cdots E_r$ is a radical extension of K. □

Theorem V.9.4

Theorem V.9.4. If F is a radical extension field of K and E is an intermediate field, then $\mathrm{Aut}_K(E)$ is a solvable group.

Proof. Let K_0 be the fixed subfield of E relative to $\mathrm{Aut}_K E$ (so $K \subset K_0 \subset E$). Then $\mathrm{Aut}_{K_0} E = \mathrm{Aut}_K E$ and the fixed field of $\mathrm{Aut}_{K_0} E$ is K_0 so E is Galois over K_0. By Exercise V.9.1, F is a radical extension of K_0 (since $K \subset K_0 \subset E \subset F$; F is radical over K and so is radical over intermediate fields by the Exercise). By the definition of radical extension, F is then algebraic over K_0 and so E is algebraic over K. Our goal is to show that $\mathrm{Aut}_K E$ is a solvable group. However, $\mathrm{Aut}_K E = \mathrm{Aut}_{K_0} E$ where E is algebraic and Galois over K_0; so WLOG we can assume that E is algebraic and Galois over K to begin with.
Let N be a normal closure of F over K. By Lemma V.9.3, N is a radical extension of K. Since $K \subset E \subset F$ where E is algebraic and Galois over K (WLOG as above), then by Lemma V.2.13, E is stable (relative to F and K). That is, every K-automorphism in $\mathrm{Aut}_K F$ maps E to itself.

Theorem V.9.4 (continued 1)

Proof (continued). Consequently, for any $\sigma \in \mathrm{Aut}_K N$ we can restrict σ to E (i.e., $\sigma|_E$) to produce an element of $\mathrm{Aut}_K E$. Let $\theta : \mathrm{Aut}_K N \to \mathrm{Aut}_K E$ be defined as $\theta(\sigma) = \sigma|_E$. Then θ is a homomorphism because $\theta(\sigma_1\sigma_2) = (\sigma_1\sigma_2)|_E = \sigma_1|_E\sigma_2|_E = \theta(\sigma_1)\theta(\sigma_2)$. Now since N is normal over K, then N is a splitting field over K by Theorem V.3.14 (the (i)⇒(ii) part), and so N is a splitting field over E. Now for $\sigma \in \mathrm{Aut}_K E$ we know that $\sigma : E \to E$ is an isomorphism and since N is a splitting field of E, then by Theorem V.3.8, σ can be extended to an isomorphism mapping $N \to N$. That is, σ extends to a K-automorphism of N. Applying homomorphism θ to the extension of σ produces $\sigma \in \mathrm{Aut}_K E$. Since σ was an arbitrary element of $\mathrm{Aut}_K E$, then θ is onto (i.e., an epimorphism). Since the homomorphic image of a solvable group is solvable by Theorem II.7.11(i), if we show that $\mathrm{Aut}_K N$ is solvable then the solvability of $\mathrm{Aut}_K E$ would follow.

Theorem V.9.4 (continued 2)

Proof (continued). Let K_1 be the fixed subfield of N relative to $\mathrm{Aut}_K N = \mathrm{Aut}_{K_1} N$. Then (by definition) N is a Galois extension of K_1 and by Exercise V.9.1, N is a radical extension of K_1 since N is a radical extension of K and $K \subset K_1 \subset N$. Hence proving that $\mathrm{Aut}_K E$ is solvable can be accomplished by proving that $\mathrm{Aut}_{K_1} N$ is solvable where N is a radical extension of K_1 and N is Galois over K_1. So WLOG we may assume that F is a Galois radical extension of K.
With $F = K(u_1, u_2, \ldots, u_n)$ with $u_1^{m_1} \in K$ and $u_i^{m_i} \in K(u_1, u_2, \ldots, u_{i-1})$ for $i \geq 2$, where m_1 and m_i are chosen to be the smallest power of u_1 and u_i in $K(u_1, u_2, \ldots, u_{i-1})$. We now establish that $\mathrm{char}(K)$ does not divide m_i. This is obvious if $\mathrm{char}(K) = 0$. If $\mathrm{char}(K) = p \neq 0$ and $m_i = rp^t$ where $\gcd(r, p) = (r, p) = 1$. Then $u_i^{m-i} = u_i^{rp^t} \in K(u_1, u_2, \ldots, u_{i-1})$ and, as remarked after Definition V.9.1, u_i is a root of $x^{m_1} - u_i^{m_1} = x_i^{rp^t} - u_i^{rp^t} \in K(u_1, u_2, \ldots, u_{i-1})[x]$.

Theorem V.9.4 (continued 3)

Proof (continued). But by the Freshman's Dream (Exercise III.1.11), $x_i^{rp^t} - u_i^{rp^t} = (x_i^r - u_i^r)^{p^t}$. So the irreducible polynomial of $u^r \in F$ over $K(u_1, u_2, \ldots, u_{i-1})$ is $(x - u_i^r)^{p^t} = x^{p^t} - i_i^{rp^t} = x^{m_i} - u_i^{m_i}$ (notice that $u_i^{rp^t} = u_i^{m_i} \in K(u_1, u_2, \ldots, u_{i-1})$) and since m_i is the smallest power of u_i in $K(u_1, u_2, \ldots, u_{i-1})$ then $(x - u_i^r)^{p^t}$ is irreducible over $K(u_1, u_2, \ldots, u_{i-1})$; the "constant term" of this polynomial is $\pm$ a power of u_i). Therefore, by definition, u_i^r is purely inseparable over $K(u_1, u_2, \ldots, u_{i-1})$. But F is Galois over K (by the WLOG argument above) and so F is separable over K by Theorem V.3.11 (the (i)$\Rightarrow$(ii) part). Whence F is separable over the intermediate field $K(u_1, u_2, \ldots, u_{i-1})$ by Exercise V.3.12. So u_i^r is both separable and purely inseparable over K, and by Theorem V.6.2, $u_i^r \in K(u_1.u_2, \ldots, u_{i-1})$. So we have that $\text{char}(K) = p$ does not divide m_i, as claimed.

Theorem V.9.4 (continued 4)

Proof (continued). If $m = m_1 m_2 \cdots m_n$ (where the m_i are minimal as required in the previous paragraph) then $\text{char}(K)$ (which equals $\text{char}(F)$ by considering $1_k = 1_F$ in Theorem III.1.9(ii) for $n > 0$, and the fact that there is no $n \in \mathbb{N}$ such that all $na = 0$ for all $a \in K$ and the fact that $K \subset F$) does not divide m. Consider $x^m - 1 \in K[x]$ and let ζ be a primitive mth root of unity (which exists in the algebraic closure of K). Then $F(\zeta)$ contains all roots of $x^m - 1$ and hence is a cyclotomic extension of K. We have:

$$\begin{tikzcd}
 & F(\zeta) \arrow[dl, dash] \arrow[dr, dash] & \\
F \arrow[dr, dash] & & K(\zeta) \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

Theorem V.9.4 (continued 5)

Proof (continued). By Theorem V.8.1(ii), $F(\zeta)$ is an abelian extension of F and so (by definition of "abelian extension") is Galois over F. By Exercise V.3.15(b), $F(\zeta)$ is Galois over K (F is Galois over K WLOG as argued above, and $F(\zeta)$ is a splitting field of $x^m - 1$ over F). By the Fundamental Theorem of Galois Theory (Theorem V.2.5(ii)) we have that $\text{Aut}_K F \cong \text{Aut}_K F(\zeta)/\text{Aut}_F F(\zeta)$ (in Theorem V.2.5 we take $F = F(\zeta)$, $E = F$, $K = K$). This shows that $\text{Aut}_K F$ is the homomorphic image of $\text{Aut}_K F(\zeta)$ under canonical epimorphism (see page 43 on Section I.5). So to show that $\text{Aut}_K F$ is solvable, it is sufficient by Theorem II.7.11(i) to show that $\text{Aut}_K F(\zeta)$ is solvable. Observe that $K(\zeta)$ is an abelian (and so by the definition of Galois) extension of K by Theorem V.8.1(ii). Whence by the Fundamental Theorem of Galois Theory (Theorem V.2.5(ii) with $F = F(\zeta)$, $E = K(\zeta)$, $K = K$) we have $\text{Aut}_K K(\zeta) \cong \text{Aut}_K F(\zeta)/\text{Aut}_{K(\zeta)} F(\zeta)$. Since $\text{Aut}_K K(\zeta)$ is abelian then it is solvable trivially (see page 102).

Theorem V.9.4 (continued 6)

Proof (continued). By Theorem II.7.11(ii), if we knew that $\text{Aut}_{K(\zeta)} F(\zeta)$ were solvable, then we would know that $\text{Aut}_K F(\zeta)$ is solvable and the proof would be complete. Thus we need only prove that $\text{Aut}_{K(\zeta)} F(\zeta)$ is solvable.

As shown above, $F(\zeta)$ is Galois over K and hence, by the Fundamental Theorem of Galois Theory (Theorem V.2.5(ii)), over any intermediate field. Let $E_0 = K(\zeta)$ and define $E_i = K(\zeta, u_1, u_2, \ldots, u_i)$ for $i = 1, 2, \ldots, n$ so that $E_n = K(\zeta, u_1, u_2, \ldots, u_n) = F(\zeta)$. Let $H_i = \text{Aut}_{E_i} F(\zeta)$ be the subgroup of $\text{Aut}_{K(\zeta)} F(\zeta))$ corresponding to field E_i under Galois correspondence in the Fundamental Theorem of Galois Theory (Theorem V.2.5).

Theorem V.9.4 (continued 7)

Proof (continued). Schematically we have:

$$\begin{array}{ccccccc}
H_n & < & H_{n-1} & < \cdots < & H_i & < \cdots < & H_0 \\
\| & & \| & & \| & & \| \\
\{e\} & < & \mathrm{Aut}_{E_{n-1}}F(\zeta) & < \cdots < & \mathrm{Aut}_{E_i}F(\zeta) & < \cdots < & \mathrm{Aut}_{K(\zeta)}F(\zeta) \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
F(\zeta) = E_n & \subset & E_{n-1} & \subset \cdots \subset & E_i & \subset \cdots \subset & E_0 = K(\zeta)
\end{array}$$

Now ζ is an mth root of unity where $m = m_1 m_2 \cdots m_n$, so by Lemma V.7.10(i), $K(\zeta)$ contains a primitive m_ith root of unity for each i. Since $u_i^{m_i} \in E_{i-1}$ and $E_i = E_{i-1}(u_i)$, then by Lemma V.7.10(ii) (with $d = m_i$), E_i is a splitting field of $x^{m_i} - 1$ over E_{i-1}. By Theorem V.7.11 (the (ii)⇒(i) part), E_i is a cyclic extension of E_{i-1}; that is, $\mathrm{Aut}_{E_{i-1}}E_i$ is a cyclic group. By definition of "cyclic extension," E_i is Galois over E_{i-1}.

Theorem V.9.4 (continued 8)

Theorem V.9.4. If F is a radical extension field of K and E is an intermediate field, then $\mathrm{Aut}_K(E)$ is a solvable group.

Proof (continued). So by the Fundamental Theorem of Galois Theory (Theorem V.2.5(ii)) we have the normal subgroups $J_i \triangleleft H_{i-1}$ (or equivalently, $\mathrm{Aut}_{E_i}F(\zeta) \triangleleft \mathrm{Aut}_{E_{i-1}}F(\zeta)$) and $H_{i-1}/H_i = \mathrm{Aut}_{E_{i-1}}F(\zeta)/\mathrm{Aut}_{E_i}F(\zeta) \cong \mathrm{Aut}_{E_{i-1}}E_i$ (with $F = F(\zeta)$, $E = E_i$, $K = E_{i-1}$ in Theorem V.2.5(ii)). So $H_{i-1}/H_i \cong \mathrm{Aut}_{E_{i-1}}E_i$ is cyclic (and so abelian). Consequently, $\{e\} = H_n < H_{n-1} < \cdots < J_1 < H_0 = \mathrm{Aut}_{K(\zeta)}F(\zeta)$ is a solvable series by definition (see Definition II.8.3). By Theorem II.8.5, $\mathrm{Aut}_{K(\zeta)}F(\zeta)$ is solvable. Therefore, this result cascades back through the line of implications and WLOG's to imply that $\mathrm{Aut}_K E$ is solvable. □

Corollary V.9.5

Corollary V.9.5. Let K be a field and $f \in K[x]$. If the equation $f(x) = 0$ is solvable by radicals, then the Galois group of f is a solvable group.

Proof. If $f(x) = 0$ is solvable by radicals, then by Definition V.9.2, there is a radical extension F of K and a splitting field E of f over K such that $F \supset E \supset K$. The Galois group of f is $\mathrm{Aut}_K E$ by Definition V.4.1. By Theorem V.9.4, $\mathrm{Aut}_K E$ is a solvable group. □

Proposition V.9.6

Proposition V.9.6. Let E be a finite dimensional Galois extension field of K with solvable Galois group $\mathrm{Aut}_K(F)$. Assume that $\mathrm{char}(K)$ does not divide $[E : K]$. Then there exists a radical extension F of K such that $F \supset E \supset K$.

Proof. By the Fundamental Theorem of Galois Theory (theorem V.2.5(i)), $|\mathrm{Aut}_K E| = [E : K]$, so $\mathrm{Aut}_K E$ is a finite solvable group. By Proposition II.8.6, $\mathrm{Aut}_K E$ has a composition series whose factors are cyclic of prime order. So there is a normal subgroup H of $\mathrm{Aut}_K E$ of some prime index p; that is, $p = |(\mathrm{Aut}_K E)/H| = |\mathrm{Aut}_K E|/|H| = [E : K]/|H|$ and so $[E : K] = p|H|$. Since $\mathrm{char}(K) \nmid [E : K]$ then $\mathrm{char}(K) \nmid p$. Let $N = E(\zeta)$ be a cyclotomic extension of E where ζ is a primitive pth root of unity (which can be done by Theorem V.8.1(i)). Define $M = K(\zeta)$.

Proposition V.9.6 (continued 1)

Proof (continued). Then we have:

$$\begin{tikzcd} & N = E(\zeta) \arrow[dl, dash] \arrow[dr, dash] & \\ E \arrow[dr, dash] & & M = K(\zeta) \arrow[dl, dash] \\ & K & \end{tikzcd}$$

By Theorem V.8.1(ii), N is a finite dimensional abelian extension of E and so, by the definition of "abelian extension," N is Galois over E and, by Exercise V.3.15(b), N is Galois over K. Now $M = K(\zeta)$ is clearly a radical extension of K. If we can find a radical extension of M that contains $N = E(\zeta)$, then this extension will be radical over K by Exercise V.9.4 (since M is radical over K) and this extension will be the desired extension F.

Proposition V.9.6 (continued 2)

Proof (continued). First, observe that E is a stable intermediate field between N and K by Lemma V.2.13 (since E is Galois over K and algebraic over K by Theorem V.1.11). That is, every K-automorphism in $\mathrm{Aut}_K N$ maps E into itself. Consequently, for any $\sigma \in \mathrm{Aut}_K N$ we can restrict σ to E (i.e., $\sigma|_E$) to produce an element of $\mathrm{Aut}_K E$. Now since $K \subset M = K(\zeta)$ then $\mathrm{Aut}_M N < \mathrm{Aut}_K E$. Let $\theta : \mathrm{Aut}_M N \to \mathrm{Aut}_K E$ be defined as $\theta(\sigma) = \sigma|_E$. Then θ is a homomorphism because $\theta(\sigma_1\sigma_2) = (\sigma_1\sigma_2)|E = \sigma_1|E\sigma_2|E = \theta(\sigma_1)\theta(\sigma_2)$. If $\sigma \in \mathrm{Aut}_M N$ then $\sigma(\zeta) = \zeta$ (since $M = K(\zeta)$). If $\sigma \in \mathrm{Ker}(\theta)$ then $\sigma|E$ must be the identity and since $N = E(\zeta)$ then σ must be the identity on N. So by Theorem I.2.3(i), θ is one to one and so is a monomorphism.
We now prove the theorem by induction on $n = [E : K]$. In the case $[E : K] = 1$ we have $E = K$ and $M = K(\zeta)$ is the desired radical extension F. Assume the theorem is true for all extensions of dimension $k < n$ and consider the two possibilities:

Proposition V.9.6 (continued 3)

Proof (continued).

(i) $\mathrm{Aut}_M N$ is isomorphic under θ to a proper subgroup of $\mathrm{Aut}_K E$;

(ii) $\mathrm{Aut}_M N \cong \mathrm{Aut}_K E$ and θ is an isomorphism.

Since $\mathrm{Aut}_K E$ is solvable, then by Theorem II.7.11(i) we have that $\mathrm{Aut}_M N$ is solvable in either case. Since E is a finite dimensional extension of K by hypothesis an d$N = E(\zeta)$ is a finite dimensional extension of E (by Theorem V.1.6(iii)) then N is a finite dimensional extension of K by Theorem V.1.2. As shown above (after the diagram) N is Galois over K and so by the Fundamental Theorem of Galois Theory (Theorem V.2.5(ii)) N is Galois over the intermediate field $M = K(\zeta)$. In case (i) we have $[N : M] = |\mathrm{Aut}MN|$ and $[E : K] = |\mathrm{Aut}_K E| = n$ by Theorem V.2.15(i) and so $[N : M] < [E : K] = n$. Whence by the induction hypothesis there is a radical extension of M that contains N. As remarked in the first paragraph, this proves the theorem in case (i).

Proposition V.9.6 (continued 4)

Proof. In case (ii), let $J = \theta^{-1}(H)$. Notice that $\theta : \mathrm{Aut}_M N \to \mathrm{Aut}_K E$ is an isomorphism in this case, so $\theta^{-1} : \mathrm{Aut}_K E \to \mathrm{Aut}_M N$ is an isomorphism and since H is a normal subgroup of index p in $\mathrm{Aut}_K E$, then J is a normal subgroup of index p in $\mathrm{Aut}_M N$. Since $\mathrm{Aut}_K E$ is solvable and $\mathrm{Aut}_M N \cong \mathrm{Aut}_K E$, then $\mathrm{Aut}_M N$ is solvable and by Theorem II.7.11(i), $J < \mathrm{Aut}_M N$ is solvable. Let P be the fixed field of J relative to $\mathrm{Aut}_M N$. Then we have

$$\begin{array}{ccccc} \{\iota\} & \lhd & J = \mathrm{Aut}_P N & \lhd & \mathrm{Aut}_M N \\ \updownarrow & & \updownarrow & & \updownarrow \\ N & \supset & P & \supset & M \end{array}$$

Notice that since P is the fixed field of J and P is Galois over M by Theorem V.2.15(ii), so $J = \mathrm{Aut}_P N$. Also be Theorem V.2.5(ii) (with $F = n$, $E = P$, and $K = M$) we have $\mathrm{Aut}_M P \cong (\mathrm{Aut}_M N)/(\mathrm{Aut}_P N) = (\mathrm{Aut}_M N)/J$. But $[\mathrm{Aut}_M N : J] = p$ be construction, whence $\mathrm{Aut}_M P \cong \mathbb{Z}_p$ by Exercise I.4.3. Therefore P is a cyclic extension of $M = K(\zeta)$.

Proposition V.9.6 (continued 5)

Proposition V.9.6. Let E be a finite dimensional Galois extension field of K with solvable Galois group $\text{Aut}_K(F)$. Assume that $\text{char}(K)$ does not divide $[E:K]$. Then there exists a radical extension F of K such that $F \supset E \supset K$.

Proof (continued). By Theorem V.7.11(ii), $P = M(u)$ where u is a root of some irreducible $x^p - a \in M[x]$. Thus P is a radical extension of M where $[P:M] > 1$ and, since $[N:M] = [N:P][P:M]$ by Theorem V.1.2, then $[N:P] < [N:M] = [F:K] = n$ (since $\text{Aut}_M N \cong \text{Aut}_K E$ in case (ii)). Since $\text{Aut}_P N = J$ is solvable and N is Galois over P by Theorem V.2.5(ii), the induction hypothesis implies that there is a radical extension F of P that contains N. Since F is a radical extension of P and P is a radical extension of $M = K(\zeta)$ (and so $K(\zeta)$ is a radical extension of K), then F is a radical extension of K which contains $N = E(\zeta)$ and hence contains K. So the result holds in case (iii). □

Corollary V.9.7. Galois' Theorem

Corollary V.9.7. Galois' Theorem. Let K be a field and $f \in K[x]$ a polynomial of degree $n > 0$, where $\text{char}(K)$ does not divide $n!$ (which is always true when $\text{char}(K) = 0$). Then the equation $f(x) = 0$ is solvable by radicals if and only if the Galois group of f is solvable.

Proof. (1) Suppose $f(x) = 0$ is solvable by radicals. Then (by Definition V.9.2) there is a radical extension E of K and a splitting field E of f over K such that $F \supset E \supset K$. By Definition V.4.1, the Galois group of f is $\text{Aut}_K E$. By Theorem V.9.4, $\text{Aut}_K F$ is solvable.
(2) Suppose the Galois group of f is solvable. So let E be a splitting field of f over K (which exists since the algebraic closure of K exists by Theorem V.3.6). Then this means that $\text{Aut}_K E$ is solvable. Notice that E can be chosen to be a finite dimensional extension by Theorem V.3.2, with $[E:K] \le n!$.

Corollary V.9.7. Galois' Theorem (continued 1)

Proof (continued). By Proposition V.9.6, it is sufficient to show that E is Galois over K and $\text{char}(K) \nmid [E:K]$ (since Proposition V.9.6 then implies the existence of radical extension F of K where $F \supset E \supset K$, and then by Definition V.9.2, $f(x) = 0$ is solvable by radicals). We have hypothesized that $\text{char}(K) \nmid n!$ where n is the degree of polynomial f. By Theorem III.6.10, an irreducible factor g of f has no multiple roots in E if and only if $g' \neq 0$. Since g is a factor of f then the degree of g is between 1 and n. If $\text{char}(k) = 0$ then $g' \neq 0$ by Exercise III.6.3(a). If $\text{char}(K) = p \neq 0$ (since $\text{char}(K) \nmid n!$ then $\text{char}(K) > n$) then by Exercise III.6.3(b), $g' = 0$ if and only if g is a polynomial in x^p. But if g is a polynomial in x^p then the degree of g is greater than n, so it must be that $g' \neq 0$ in this case as well. So (by Theorem III.6.10), the irreducible factors of f are separable (see Definition V.3.10).

Corollary V.9.7. Galois' Theorem (continued 2)

Corollary V.9.7. Galois' Theorem. Let K be a field and $f \in K[x]$ a polynomial of degree $n > 0$, where $\text{char}(K)$ does not divide $n!$ (which is always true when $\text{char}(K) = 0$). Then the equation $f(x) = 0$ is solvable by radicals if and only if the Galois group of f is solvable.

Proof (continued). By Exercise V.3.13 (the (iii)⇒(ii) part), E is separable over K. Then by Theorem V.3.11 (the (ii)⇒(i) part), and the fact that E is a splitting field of f) E is Galois over K. Since $[E:K] \le n!$ then every prime that divides $[E:K]$ must also divide $n!$. Since $\text{char}(K)$ is either 0 or prime and $\text{char}(K) \nmid n!$ then $\text{char}(K) \nmid [E:K]$. As mentioned above, Proposition V.9.6 now implies that the equation $f(x) = 0$ is solvable by radicals. □