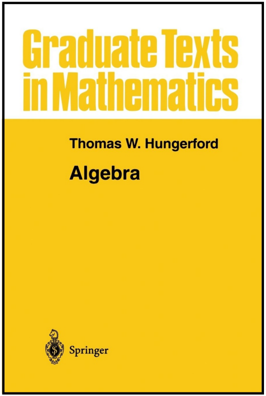
Modern Algebra

Chapter VIII. Commutative Rings and Modules

VIII.1. Chain Conditions—Proofs of Theorems



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Proof. Suppose A satisfies the minimal condition on submodules and let $A_1 \supset A_2 \supset A_3 \supset \cdots$ be an arbitrary descending chain of submodules. Then by the minimal condition hypothesis, the set $\{A_i \mid i \geq 1\}$ has a minimal element, say A_n.

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Conversely, suppose A satisfies the descending chain condition, and let S be a nonempty set of submodules of A. Then there is $B_0 \in S$. ASSUME set S has no minimal element.

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Conversely, suppose A satisfies the descending chain condition, and let S be a nonempty set of submodules of A. Then there is $B_0 \in S$. ASSUME set S has no minimal element. Then for each submodule B in S there exists at least one submodule B' in S such that $B' \subset B$ and $B' \neq B$. We use the Axiom of Choice to choose for each $B \in S$ one such B'.

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Theorem VIII.1.4 (condition)

Proof (continued). Define $f : S \to S$ by $f(B) = B'$ in this notation (notice that f is then a choice function). By Theorem 0.6.2, The Recursion Theorem, with $f_n = f$ for all $n \in \mathbb{N}$ there is a function $\varphi : \mathbb{N} \cup \{0\} \to S$ such that $\varphi(0) = B_0$ and $\varphi(n+1) = f(\varphi(n)) = \varphi(n)'$ (The Recursion Theorem allows us to create a chain of modules). Denote $\varphi(n) = B_n$ so that $\varphi(n+1) = B_{n+1} = f(\varphi(n)) = f(B_n) = B_n'$. Then we have the descending chain $B_0 \supset B_0' = B_1 \supset B_1' = B_2 \supset \cdots$ where $B_i \neq B_{i+1}$ for all $i \in \mathbb{N} \cup \{0\}$.

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The proof for ascending chains and the maximum condition is similar and is left as Exercise VIII.1.A. □

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Proof (continued). Define $f : S \to S$ by $f(B) = B'$ in this notation (notice that f is then a choice function). By Theorem 0.6.2, The Recursion Theorem, with $f_n = f$ for all $n \in \mathbb{N}$ there is a function $\varphi : \mathbb{N} \cup \{0\} \to S$ such that $\varphi(0) = B_0$ and $\varphi(n+1) = f(\varphi(n)) = \varphi(n)'$ (The Recursion Theorem allows us to create a chain of modules). Denote $\varphi(n) = B_n$ so that $\varphi(n+1) = B_{n+1} = f(\varphi(n)) = f(B_n) = B_n'$. Then we have the descending chain $B_0 \supset B_0' = B_1 \supset B_1' = B_2 \supset \cdots$ where $B_i \neq B_{i+1}$ for all $i \in \mathbb{N} \cup \{0\}$. But this is a descending chain that does not satisfy the descending chain condition, CONTRADICTING the hypothesis that A satisfies the descending chain condition. So the assumption that set S has no minimal element is false, and so module A satisfies the minimal condition on submodules.

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Theorem VIII.1.5

Theorem VIII.1.5. Let $\{0\} \to A \xrightarrow{f} B \xrightarrow{g} C \to \{0\}$ be a short exact sequence of modules. Then B satisfies the ascending (respectively, descending) chain condition on submodules if and only if A and C satisfy it.

Proof. Suppose B satisfies the ascending chain condition. Since $f(A)$ is a submodule of B (the homomorphic image of a module is a module; see the example after Definition IV.1.3) then $f(A)$ also satisfies the ascending chain condition (any ascending chain of submodules of $f(A)$ is also an ascending chain of submodules of B). Since f is one to one then A is isomorphic to $f(A)$, so A also satisfies the ACC.

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Theorem VIII.1.5 (continued 1)

Proof (continued). Now suppose A and C satisfy the ACC. Let $B_1 \subset B_2 \subset B_3 \subset \cdots$ be an ascending chain of submodules of B. For each $i \in \mathbb{N}$ let $A_i = f^{-1}(f(A) \cap B_i)$ and $C_i = g(B_i)$. Let $f_i = f|_{A_i}$ and $g_i = g|_{B_i}$ (restrictions of f and g).

We now show that $\{0\} \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to \{0\}$ is a short exact sequence for each $i \in \mathbb{N}$; that is, we show $\mathrm{Ker}(f_i) = \{0\}$, $\mathrm{Im}(f_i) = \mathrm{Ker}(g_i)$, and $\mathrm{Im}(g_i) = C_i$ for $i \in \mathbb{N}$.

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$$f_i(A_i) = f|_{A_i}(A_i) = f(A_i) = f(f^{-1}(f(A) \cap B_i)) = f(A) \cap B_i.$$

Since $\text{Ker}(f) = \{0\}$ then $\text{Ker}(f_i) = \text{Ker}(f_i|_{A_i}) = \{0\}$ for each $i \in \mathbb{N}$. Also, we have $\text{Im}(f_i) = f(A) \cap B_i$. Now $g_i(B_i) = g|_{B_i}(B_i) = g(B_i) = C_i$ (so g_i is onto). In the given short exact sequence, $\text{Im}(f) = \text{Ker}(g)$, so

$$\text{Ker}(g_i) = \text{Ker}(g|_{B_i}) = \text{Ker}(g) \cap B_i = \text{Im}(f) \cap B_i = f(A) \cap B_i = \text{Im}(f_i)$$

for all $i \in \mathbb{N}$.

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Proof (continued). Finally, $C_i = g(B_i)$ by the definition of C_i, so $\text{Im}(g_i) = C_i$ for $i \in \mathbb{N}$. Therefore $\{0\} \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to \{0\}$ is a short exact sequence.

We now claim $A_1 \subset A_2 \subset A_3 \subset \cdots$ and $C_1 \subset C_2 \subset C_3 \subset \cdots$. Since $B_1 \subset B_2 \subset B_3 \subset \cdots$ by hypothesis then $f(A) \cap B_i \subset f(A) \cap B_{i+1}$ and $A_i = f^{-1}(f(A) \cap B_i) \subset f^{-1}(f(A) \cap B_{i+1}) = A_{i+1}$ for $i \in \mathbb{N}$. Also, $B_i \subset B_{i+1}$ implies $C_i = g(B_i) \subset g(B_{i+1}) = C_{i+1}$ for $i \in \mathbb{N}$.

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Since in this case we hypothesize that A and C satisfy the ACC, then there is $n \in \mathbb{N}$ such that $A_i = A_n$ and $C_i = C_n$ for all $i \geq n$. For each $i \geq n$ consider the commutative diagram...

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Theorem VIII.1.5 (continued 3)

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$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta_i & & \downarrow \gamma & & \\ 0 & \longrightarrow & A_i & \xrightarrow{f_i} & B_i & \xrightarrow{g_i} & C_i & \longrightarrow & 0 \end{array}$$

where α and γ are identity maps (since $A_i = A_n$ and $B_i = B_n$) and β_i is the inclusion map (since $B_n \subset B_i$ for $i \geq n$). By the Short Five Lemma, Lemma IV.1.17, β_i is a one to one and onto isomorphism (since α and γ are). So $B_n = B_i$ and this holds for all $i \geq n$. Since $B_i \subset B_2 \subset B_3 \subset \cdots$ is an arbitrary chain of submodules of B, then B satisfies the ACC, as claimed.

Theorem VIII.1.5 (continued 3)

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The proof for the descending chain condition is similar and left as Exercise VIII.1.B. □

Theorem VIII.1.5 (continued 3)

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Corollary VIII.1.6

Corollary VIII.1.6. If A is a submodule of a module B, then B satisfies the ascending (respectively, descending) chain condition if and only if A and B/A satisfy it.

Proof. Consider the sequence $\{0\} \to A \xrightarrow{f} B \xrightarrow{g} B/A \to \{0\}$, where f is the inclusion map and g is the canonical epimorphism, so that $g(b) = b + A$. Then $\text{Ker}(f) = \{0\}$, $\text{Im}(f) = A = \text{Ker}(g)$, and $\text{Im}(g) = B/A$ so that this is a short exact sequence. By Theorem VIII.1.5, B satisfies the ACC (respectively, DCC) if and only if both A and B/A satisfy it. □

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Corollary VIII.1.7

Corollary VIII.1.7. If $A_1, A_2, \ldots, A_n$ are modules, then the direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ satisfies the ascending (respectively, descending chain condition on submodules if and only if each A_i satisfies it.

Proof. We prove by induction. For $n = 2$, consider the sequence $\{0\} \to A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to \{0\}$ where ι_1 is a canonical injection and π_2 is a canonical projection. Then $\mathrm{Ker}\iota_i) = \{0\}$, $\mathrm{Im}(\iota_1) = A_1 \oplus \{0\} = \mathrm{Ker}(\pi_2)$, and $\mathrm{Im}(\pi_2) = A_2$ so that this is a short exact sequence.

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Now suppose the result holds for $n = k$ and consider $\{0\} \to A_1 \oplus A_2 \oplus \cdots \oplus A_k \overset{\iota}{\to} A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus A_{k+1} \overset{\pi_{k+1}}{\to} \{0\}$ where ι is the canonical injection of $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ into $A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus A_{k+1}$ and π_{k+1} is the canonical projection.

Corollary VIII.1.7

Corollary VIII.1.7. If $A_1, A_2, \ldots, A_n$ are modules, then the direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ satisfies the ascending (respectively, descending chain condition on submodules if and only if each A_i satisfies it.

Proof. We prove by induction. For $n = 2$, consider the sequence $\{0\} \to A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to \{0\}$ where ι_1 is a canonical injection and π_2 is a canonical projection. Then $\mathrm{Ker}\iota_i) = \{0\}$,
$\mathrm{Im}(\iota_1) = A_1 \oplus \{0\} = \mathrm{Ker}(\pi_2)$, and $\mathrm{Im}(\pi_2) = A_2$ so that this is a short exact sequence. By Theorem VIII.1.5, $A_1 \oplus A_2$ satisfies the ACC (respectively, DCC) if and only if A_1 and A_2 satisfy it. The result holds for $n = 2$.

Now suppose the result holds for $n = k$ and consider $\{0\} \to A_1 \oplus A_2 \oplus \cdots \oplus A_k \xrightarrow{\iota} A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus A_{k+1} \xrightarrow{\pi_{k+1}} \{0\}$ where ι is the canonical injection of $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ into $A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus A_{k+1}$ and π_{k+1} is the canonical projection.

Corollary VIII.1.7 (continued)

Corollary VIII.1.7. If $A_1, A_2, \ldots, A_n$ are modules, then the direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ satisfies the ascending (respectively, descending chain condition on submodules if and only if each A_i satisfies it.

Proof (continued). As argued above, this is a short exact sequence. Applying Theorem VIII.1.5, $A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus A_{k+1}$ satisfies the ACC (respectively, DCC) if and only if $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ (and, by the induction hypothesis, $A_1, A_2, \ldots, A_k$) and A_{k+1} satisfy it. So the result holds for $n = k + 1$. The general result now follows by induction. □

Theorem VIII.1.8

Theorem VIII.1.8. If R is a left Noetherian (respectively, Artinian) ring with identity, then every finitely generated unitary left R-module A satisfies the ascending (respectively, descending) chain condition on the submodules. This also holds if "left" is replaced with "right."

Proof. Suppose A is a finitely generated unitary left R-module where R is left Noetherian. Then by Corollary IV.2.2 there is a finitely generated free R-module F and an onto homomorphism (i.e., an epimorphism) $\pi : F \to A$.

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Proof. Suppose A is a finitely generated unitary left R-module where R is left Noetherian. Then by Corollary IV.2.2 there is a finitely generated free R-module F and an onto homomorphism (i.e., an epimorphism) $\pi : F \to A$. Since F is finitely generated then it has a finite basis. By by Theorem IV.2.1, f is a direct sum of a finite number of copies of R. Then by Corollary VIII.1.7, F is left Noetherian (respectively, left Artinian) since R is. By Theorem IV.1.7 (the "in particular" part), $A \cong F/\text{Ker}(\pi)$.

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Theorem VIII.1.9

Theorem VIII.1.9. A module A satisfies the ascending chain condition on submodules if and only if every submodule of A is finitely generated. In particular, a commutative ring R is Noetherian if and only if every ideal of R is finitely generated.

Proof. Suppose A satisfies the ACC on submodules. Let B be a submodule of A. Let S be the set of all finitely generated submodules of B.

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Proof. Suppose A satisfies the ACC on submodules. Let B be a submodule of A. Let S be the set of all finitely generated submodules of B. Now $\{0\} \in S$ so S is nonempty and so by Theorem VIII.1.4 S satisfies the maximum condition. Hence there is a maximal element C. Since $C \in S$ then C is finitely generated by $c_1, c_2, \ldots, c_n$.

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Theorem VIII.1.9 (continued)

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Proof (continued). Now suppose every submodule of A is finitely generated. Let $A_1 \subset A_2 \subset A_3 \subset \cdots$ be an ascending chain of submodules of A. "It is easy to verify" that $\cup_{i=1}^{\infty} A_i$ is a submodule of A (Hungerford claims on page 375) and so finitely generated by hypothesis. Say $\cup_{i=1}^{\infty} A_i$ is generated by $a_1, a_2, \ldots, a_k$. Since each a_i is in some A_j, there is an index n such that $a_i \in A_n$ for $i = 1, 2, \ldots, k$. So $\{a_1, a_2, \ldots, a_k\} \subset A_n$ and $\cup_{i=1}^{\infty} A_i \subset A_n$. Whence $A_i = A_n$ for $i \geq n$ and, since $A_1 \subset A_2 \subset A_3 \subset \cdots$ is an arbitrary ascending chain of submodules, then A satisfies the ACC on submodules and the converse claim holds. □

Theorem VIII.1.9 (continued)

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Theorem VIII.1.11

Theorem VIII.1.11. A nonzero module A has a composition series if and only if A satisfies both the ascending and descending chain conditions on submodules.

Proof. First, suppose A has a composition series S of length n. ASSUME at least one of the chain conditions fails to hold. Then there are submodules $A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1}$, where $A_1 \neq A_{i+1}$ for $i = 0, 1, 2, \ldots, n$, which form a normal series T of length $n + 1$ (since the chain could not "end" at A_n). By Theorem VIII.1.10(a), normal series S and T have refinements that are equivalent.

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Theorem VIII.1.11 (continued 1)

Proof (continued). So the assumption that at least one of the chain conditions does not hold is false and hence both chain conditions must hold, as claimed.

Now suppose both chain conditions hold. Let B be a nonzero submodule of A and let $S(B)$ be the set of all submodules C of B such that $C \neq B$. So if B has no proper submodules then $S(B) \neq \{0\}$. Also define $S(\{0\}) = \{0\}$.

Theorem VIII.1.11 (continued 1)

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Theorem VIII.1.11 (continued 1)

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Theorem VIII.1.11 (continued 1)

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Theorem VIII.1.11 (continued 2)

Theorem VIII.1.11. A nonzero module A has a composition series if and only if A satisfies both the ascending and descending chain conditions on submodules.

Proof (continued). Notice that $\varphi(n+1)$ is a maximal (submodule of $\varphi(n)$) element of $S(\varphi(n))$ (this is what the prime notation represents), so $\varphi(n+1) \subset \varphi(n)$ and $A \supset A_1 \supset A_2 \supset \cdots$ is a descending chain of submodules of A. Since A satisfies the DCC then there is $n \in \mathbb{N}$ such that $A_i = A_n$ for $i \geq n$. Since $\varphi(n+1)$ is a submodule of $\varphi(n)$ but $\varphi(n+1) \neq \varphi(n)$ (though we allow $\varphi(n+1) = \{0\}$), then the only way that $\varphi(n+1) = \varphi(n)$ is when $\varphi(n) = \{0\}$. That is, $A_{n+1} = A_n$ if and only if $A_n = A_{n+1} = \{0\}$.

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Proof (continued). Notice that $\varphi(n+1)$ is a maximal (submodule of $\varphi(n)$) element of $S(\varphi(n))$ (this is what the prime notation represents), so $\varphi(n+1) \subset \varphi(n)$ and $A \supset A_1 \supset A_2 \supset \cdots$ is a descending chain of submodules of A. Since A satisfies the DCC then there is $n \in \mathbb{N}$ such that $A_i = A_n$ for $i \geq n$. Since $\varphi(n+1)$ is a submodule of $\varphi(n)$ but $\varphi(n+1) \neq \varphi(n)$ (though we allow $\varphi(n+1) = \{0\}$), then the only way that $\varphi(n+1) = \varphi(n)$ is when $\varphi(n) = \{0\}$. That is, $A_{n+1} = A_n$ if and only if $A_n = A_{n+1} = \{0\}$. Let m be the smallest index such that $A_m = \{0\}$. Then $m \leq n$ and $A_k \neq \{0\}$ for all $k < m$. Furthermore, for each $k < m$, A_{k+1} is a maximal submodule of A_k such that $A_{k+1} \neq A_k$.

Theorem VIII.1.11 (continued 2)

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Theorem VIII.1.11 (continued 3)

Theorem VIII.1.11. A nonzero module A has a composition series if and only if A satisfies both the ascending and descending chain conditions on submodules.

Proof (continued). So each A_k/A_{k+1} is nonzero (since $A_{k+1} \neq A_k$) and has no proper submodules by Theorem IV.1.10 (since $C_k/A_{k+1} \subset A_k/A_{k+1}$ implies C_k is a submodule of A_k which contains A_{k+1} by Theorem IV.1.10, but A_{k+1} as a maximal submodule of A_k not equal to A_k, so we must have $C_k = A_k$ and $C_k/A_{k+1} = A_k/A_{k+1}$; hence no proper submodules of A_k/A_{k+1}; this is whey we defined $S(B)$ as the set of all submodules of B not equal to B). Therefore, $A = A_1 \supset A_2 \supset \cdots \supset A_m = \{0\}$ is a composition series of $A$and the second claim follows. □

Corollary VIII.1.12

Corollary VIII.1.12. If D is a division ring, then the ring $\text{Mat}_n(D)$ of all $n \times n$ matrices over D is both Artinian and Noetherian.

Proof. To show ring $\text{Mat}_n(D)$ is both Artinian and Noetherian, we need to show that it satisfies both the ACC and DCC on ideals, by Definition VIII.1.2. If we interpret $R = \text{Mat}_n(D)$ as a R-module (so that the ideals are submodules), then by Theorem VIII.1.11 is suffices to show that $\text{Mat}_n(D)$ has a composition series of left R-modules (to cover the conditions of left Noetherian and left Artinian). For each $i \in \{1, 2, \ldots, n\}$ let $e_i \in R$ be the matrix with 1_D in position (i, i) and 0 elsewhere.

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We now claim that $Re_i = \{Ae_i \mid A \in R\}$ is a left ideal (and so a submodule) of R consisting of all matrices in R with column j zero for all $j \neq i$. Because of the row times column definition of matrix multiplication, Ae_i is an $n \times n$ matrix with column i the same as the ith column of A and all other columns of all 0's. So Re_i consists of all such matrices as described and only those matrices.

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Corollary VIII.1.12 (continued 1)

Proof (continued). Next, we claim Re_i is a minimal nonzero left ideal (that is, Re_i has no proper submodules). If there is a nonzero submodule of Re_i then it contains an $n \times n$ matrix $M_1 \in Re_i$ with some nonzero entry in position (j, i), say $r_{ji} \neq 0$. Notice that the three elementary row operations are defined in Definition VII.2.7 and that over a ring with identity (since D is a division ring so it has identity), each elementary row operation can be performed by multiplication on the left by an elementary matrix (see also Definition VII.2.7) by Theorem VII.2.8 (and by common knowledge from linear algebra). Since $r_{ji} \neq 0$ then we can perform the elementary row operation by multiplying row j of M_1 by unit $r(r_{ji})^{-1}$ for any nonzero $r \in D$ to produce entry $r \neq 0$ in the (j, i) position of a new matrix M_r. Notice that we are using the fact that D is an integral domain here! This row operation can be performed by multiplication on the left by the appropriate elementary matrix in $\mathrm{Mat}_n(D)$, by Theorem VII.2.8, so M_r must be in Re_i since it is an R-module.

Corollary VIII.1.12 (continued 1)

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Corollary VIII.1.12 (continued 2)

Proof (continued). Consider the entry r_{ki} in position (k, i) of M_r. We can perform the elementary row operation of adding to row k unit $(s - r_{ki})(r_{ij})^{-1}$ times row i to produce an entry of S in position (k, i) for any $x \in D$ is a new matrix M_s (notice that $(s - r_{ki})(r_{ji})^{-1}$ is a nonunit if and only if $x = r_{ki}$, in which case there is no need to perform this row operation). Similarly, by Theorem VII.2.8, $M_s \in Re_i$. So Re_i must contain a matrix with position (j, i) having entry $r \in D$ and with position (k, i) having entry $s \in D$, and this can be done for any $r, s \in D$. Since k was arbitrary, then we can make position (k, i) for $1 \leq k \leq n$ any entry we desire through elementary row operations and so a nonzero submodule of Re_i must contain all matrices with all columns 0 except column i. That is, Re_i has no nonzero submodules and hence Re_i is a maximal submodule of $\text{Mat}_n(D)$.

Corollary VIII.1.12 (continued 2)

Proof (continued). Consider the entry r_{ki} in position (k, i) of M_r. We can perform the elementary row operation of adding to row k unit $(s - r_{ki})(r_{ij})^{-1}$ times row i to produce an entry of S in position (k, i) for any $x \in D$ is a new matrix M_s (notice that $(s - r_{ki})(r_{ji})^{-1}$ is a nonunit if and only if $x = r_{ki}$, in which case there is no need to perform this row operation). Similarly, by Theorem VII.2.8, $M_s \in Re_i$. So Re_i must contain a matrix with position (j, i) having entry $r \in D$ and with position (k, i) having entry $s \in D$, and this can be done for any $r, s \in D$. Since k was arbitrary, then we can make position (k, i) for $1 \leq k \leq n$ any entry we desire through elementary row operations and so a nonzero submodule of Re_i must contain all matrices with all columns 0 except column i. That is, Re_i has no nonzero submodules and hence Re_i is a maximal submodule of $\text{Mat}_n(D)$.

Corollary VIII.1.12 (continued 3)

Proof (continued). Let $M_0 = \{0\}$ and for $i \geq 1$ let $M_i = R(e_1 + e_2 + \cdots + e_i)$. Then M_i includes all $n \times n$ matrices with only zeros in columns $i+1, i+2, \ldots, m$. Similar to the argument above, M_i is a left ideal of R. Consider M_i/M_{i-1}. The elements of M_i/M_{i-1} are of the form $m + M_{i-1}$ where $m \in M_i$. So $m + M_{i-1} = n + M_{i-1}$ if and only if the ith column of m is the same as the ith column of n. Define $f : M_i/M_{i-1} \to Re_i$ mapping $m + M_{i-1}$ to the matrix in Re_i with ith column the same as m. Then f is one to one, onto, and (since addition and multiplication of cosets is performed by representatives) a ring homomorphism. That is, $M_i/M_{i-1} \cong Re_i$.

Corollary VIII.1.12 (continued 3)

Proof (continued). Let $M_0 = \{0\}$ and for $i \geq 1$ let $M_i = R(e_1 + e_2 + \cdots + e_i)$. Then M_i includes all $n \times n$ matrices with only zeros in columns $i+1, i+2, \ldots, m$. Similar to the argument above, M_i is a left ideal of R. Consider M_i/M_{i-1}. The elements of M_i/M_{i-1} are of the form $m + M_{i-1}$ where $m \in M_i$. So $m + M_{i-1} = n + M_{i-1}$ if and only if the ith column of m is the same as the ith column of n. Define $f : M_i/M_{i-1} \to Re_i$ mapping $m + M_{i-1}$ to the matrix in Re_i with ith column the same as m. Then f is one to one, onto, and (since addition and multiplication of cosets is performed by representatives) a ring homomorphism. That is, $M_i/M_{i-1} \cong Re_i$. So $R = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset M_0 = \{0\}$ is a composition series (since $M_i/M_{i-1} \cong Re_i$ has no proper submodules) of left R-modules. A similar argument with right ideals $e_iR = \{e_iA \mid A \in R\}$ (consisting of all matrices in $R = \text{Mat}_n(D)$ with row j zero for all $j \neq i$). □

Corollary VIII.1.12 (continued 3)

Proof (continued). Let $M_0 = \{0\}$ and for $i \geq 1$ let $M_i = R(e_1 + e_2 + \cdots + e_i)$. Then M_i includes all $n \times n$ matrices with only zeros in columns $i+1, i+2, \ldots, m$. Similar to the argument above, M_i is a left ideal of R. Consider M_i/M_{i-1}. The elements of M_i/M_{i-1} are of the form $m + M_{i-1}$ where $m \in M_i$. So $m + M_{i-1} = n + M_{i-1}$ if and only if the ith column of m is the same as the ith column of n. Define $f : M_i/M_{i-1} \to Re_i$ mapping $m + M_{i-1}$ to the matrix in Re_i with ith column the same as m. Then f is one to one, onto, and (since addition and multiplication of cosets is performed by representatives) a ring homomorphism. That is, $M_i/M_{i-1} \cong Re_i$. So $R = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset M_0 = \{0\}$ is a composition series (since $M_i/M_{i-1} \cong Re_i$ has no proper submodules) of left R-modules. A similar argument with right ideals $e_iR = \{e_iA \mid A \in R\}$ (consisting of all matrices in $R = \mathrm{Mat}_n(D)$ with row j zero for all $j \neq i$). □