Modern Algebra 1, MATH 5420

Homework 1, Section III.1

Due Friday, January 29 at 1:40

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses.

III.1.3. A ring R such that $a^2 = a$ for all $a \in R$ is a Boolean ring. Prove that every Boolean ring R is commutative and $a + a = 0$ for all $a \in R$. HINT: Consider $a + b = (a + b)^2$ and use this to show that $-ab = ab$ and $-ab = ba$.

III.1.11 (The Freshman's Dream) Let R be a commutative ring with identity of prime characteristic p. Prove that if $a, b \in R$ then $(a \pm b)^{p^n} = a^{p^n} \pm b^{p^n}$ for all integers $n \geq 0$. Prove the result for $n = 1$ and then prove using mathematical induction. HINT: First prove for $(a + b)$, then replace b with $-b$. Notice that if $p = 2$ then $b + b = 0$ and so $b = -b$.

III.1.12. An element of a ring R is *nilpotent* if $a^n = 0$ for some n.

(a) Prove that in a commutative ring, if a and b are nilpotent, then $(a + b)$ is nilpotent. HINT: Suppose $a^n = 0$ and $b^m = 0$ and consider $(a + b)^{n+m}$. Notice that the Binomial Theorem (Theorem III.1.6) holds in any commutative ring—we only need to omit the terms a^0 and b^0 in the $k = 0$ term and the $k = n$ term.

III.1.A. Prove that the set of units in a ring with identity form a group under multiplication.