Section 0.7. The Axiom of Choice, Order, and Zorn's Lemma

Note. In this section, we state the Axiom of Choice and two statements equivalent to it (Zorn's Lemma and The Well-Ordering Principle). We state some implications of these axioms and give an idea as to why these axioms are desirable.

Note. If $I \neq \varnothing$ and $\{A_i \mid i \in I\}$ is a family of sets such that $A_j = \varnothing$ for some $j \in I$ then $\prod_{i \in I} A_i = \varnothing$ since there can be no function $f : I \to \cup A_i$ such that $f(j) \in A_j = \varnothing$. So if $A_i \neq \varnothing$ for all $i \in I$, we expect $\prod_{i \in I} A_i \neq \varnothing$. However it can be proved that this is independent of the usual (Zermelo-Frankel) axioms of set theory. However, this is consistent with the ZF axioms of set theory (Hungerford references Paul Cohen's *Set Theory and the Continuum Hypothesis*, 1966). This (and other issues to be mentioned) leads us to ASSUME AS AN AXIOM the following.

The Axiom of Choice. The product of a family of nonempty sets indexed by a nonempty set is nonempty.

Note. An alternate version of the Axiom of Choice is given in Exercise 0.7.4: Let S be a set. A *choice function* for S is a function f from the set of all nonempty subsets of S to S such that $f(A) \in A$ for all $A \neq \varnothing$, $A \subset S$. The Axiom of Choice is equivalent to the claim: Every set S has a choice function.

Note. The Zermelo-Frankel axioms with the Axiom of Choice are often abbreviated "ZFC."

Definition. A *partially ordered set* is a nonempty set A together with a relation R on $A \times A$ (called a *partial ordering*) of A which is reflexive, transitive and *anit-symmetric*. Antisymmetry means that if $(a, b), (b, a) \in R$ then $a = b$. For partial ordering R, when $(a, b) \in R$ we denote this as $a \leq b$. Elements $a, b \in A$ are *comparable* if either $a \leq b$ or $b \leq a$. A partial ordering of a set A such that any two elements are comparable is called a *total ordering* (or *linear ordering*).

Example. Let A be the power set of $\{1, 2, 3, 4, 5\}$ and define $C \leq D$ if $C \subset D$. This is a partial ordering, but not a total ordering since, for example, $\{1, 2\}$ and $\{2, 3\}$ are not comparable.

Definition. Let A be partially ordered by $\leq$. An element $a \in A$ is *maximal* in A if every $c \in A$ which is comparable to a, we have $c \leq a$. An *upper bound* of a nonempty subset B of A is an element $d \in A$ such that $b \leq d$ for every $b \in B$. A nonempty subset B of A that is totally ordered by $\leq$ is a *chain* in A.

Note. Under the Zermelo-Frankel axioms, the following is equivalent to the Axiom of Choice:

Zorn's Lemma. If A is a nonempty partially ordered set such that every chain in A has an upper bound in A, then A contains a maximal element.

Hungerford references Hewitt and Stromberg's *Real Analysis and Abstract Analysis* (1969) for the claim that the Axiom of Choice and Zorn's Lemma are equivalent.

Note. In Introduction to Modern Algebra 2 (MATH 4137/5137) Zorn's Lemma is used in the proof that: Every field F has an algebraic closure $\overline{F}$. See my online notes for Introduction to Modern Algebra 2 on VI.31. Algebraic Extensions.

Definition. Let B be a nonempty subset of a partially ordered set A (under $\leq$). an element $c \in B$ is a *least element* of B provided $c \leq b$ for all $b \in B$. If every nonempty subset of A has a least element, then A is *well ordered*.

Example. Notice that the natural numbers are well ordered under the usual less than or equal to.

Note. Another statement equivalent to the Axiom of Choice (assuming Zermelo-Frankel) is:
The Well Ordering Principle. If A is a nonempty set, then there exists a total ordering $\leq$ of A such that A is well ordered under $\leq$.

Note. The Axiom of Choice axiomatized the existence of certain sets without giving an explicit description of how the sets are constructed (that is, what the choice function "chooses" from set A_i). In Real Analysis 1 (MATH 5210) you will see the "construction" of a nonmeasurable subset of the interval $[0, 1)$; see my online Real Analysis 1 notes on 2.6. Nonmeasurable Sets (Roydens 3rd Edition). The set is given by the Axiom of Choice and, as a result, you do not know any element of $[0, 1)$ which is necessarily in the set. There is no element in $[0, 1)$ that is necessarily NOT in the set. You can show that the set is uncountable and contains exactly one rational number—but that is about it!

Note. The Axiom of Choice is also used in the Banach-Tarski Paradox. In this, a sphere of radius 1 is partitioned into pieces. Some of the pieces are rigidly put together to form a first sphere of radius 1 and the rest of the pieces are rigidly put together to form a second sphere of radius 1. So it seems that volume has come out of nowhere! This is another reason that some mathematicians object to the Axiom of Choice. (The resolution to the Banach -Tarski Paradox is that the pieces are of such strange shapes that they do not have a definable volume [and they are not volume 0]. So the pieces do not behave in the sense that the sum of the volume of the pieces equals the volume of the sphere from which they came. This is related to the nonmeasurable set mentioned above.) See the supplement I use in Real Analysis 1 (MATH 5210) on Nonmeasurable Sets and the Banach-Tarski Paradox.

Note. My favorite reference for set theory is Hrbacek and Jech's *Introduction to Set Theory*, 2nd Edition (1984). Much of the following information is from this source. I have some online notes based on this source.

Note. The Banach-Tarski Paradox is probably the best known (and most objectionable) result which uses the Axiom of Choice. Some other results which require the Axiom of Choice (and which are more "useful") include the following:

- Every infinite set has a countable subset.
- For any sets A and B, either $|A| \leq |B|$ or $|B| \leq |A|$.
- The union of a countable collection of countable sets is countable.

- The set of all real numbers is not the union of countably many countable sets.
- The ε-δ definition of "$f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = a$" is equivalent to the claim that for every sequence $\{x_n\} \subset \mathbb{R}$ such that $\{x_n\} \to a$ we have $\{f(x_n)\} \to f(a)$.
- Every vector space has a basis.
- Every field F has an algebraic closure $\overline{F}$.

Note. The fact that every infinite set has a countable subset, allows us to conclude that there is a "smallest" level of infinity. Since for any sets A and B we have either $|A| \leq |B|$ or $|B| \leq |A|$, then we can discuss the cardinality of infinite sets and always compare the cardinalities of two sets.

Note. The fact that every vector space has a basis is rather a long story. It is best explored in our class Introduction to Functional Analysis (MATH 5740). This is addressed in Hong, Wang, and Gardner's *Real Analysis with an Introduction to Wavelets and Applications* (2005). So my online notes on 5.1. Groups, Fields, and Vector Spaces.

Note. Hrbacek and Jech comment on page 180 that "...the irreplaceable role of the Axiom of Choice is to simplify general topological and algebraic considerations which otherwise would be bogged down in irrelevant set-theoretic detail. For this pragmatic reason, we expect that the Axiom of Choice will always keep its place in set theory."

Note. The Well Ordering Principle allows us to extend the Principle of Mathematical Induction (Theorem 0.6.1) to any well ordered set.

Theorem 0.7.1. Principle of Transfinite Induction.
If B is a subset of a well ordered set A such that for every $a \in A$,

$$\{c \in A \mid c < a\} \subset B \text{ imples } a \in B$$

then $A = B$.

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