Section 0.8. Cardinal Numbers

Note. In this section, we consider a topic from set theory concerning the cardinalities of infinite sets. A more detailed coverage of this topic can be found in the readable text by K. Hrbacek and T. Jech "Introduction to Set Theory," 2nd Edition Revised and Expanded, in Pure and Applied Mathematics, A Series of Monographs and Textbooks, Marcel Dekker (1984); I have some online notes from this source.

Note. The "standard" axioms of set theory are called the Zermelo-Frankel Axioms with Choice, or "ZFC Axioms" for short. Quoting from Hrbacek and Jech (Chapter 12, The Axiomatic Set Theory) we have:

Axiom of Existence. There exists a set which has no elements.

Axiom of Extensionality. If every element of X is an element of Y and every element of Y is an element of X, then $X = Y$.

Axiom Schema of Comprehension. Let $P(x)$ be a property of x. For any A, there is a B such that $x \in B$ is and only if $x \in A$ and $P(x)$ holds.

Axiom of Pair. For any A and B, there is C such that $x \in C$ if and only if $x = A$ or $x = B$.

Axiom of Union. For any S, there is U such that $x \in U$ if and only if $x \in A$ for some $A \in S$.

Axiom of Power Set. For any S, there is P such that $X \in P$ if and only if $X \subseteq S$.

Axiom of Infinity. An inductive set exists.

Axiom Schema of Replacement. let $P(x, y)$ be a property such that for every x there is a unique y for which $P(x, y)$ holds. For every A there is B such that for every $x \in A$ there is $y \in B$ for which $P(x, y)$ holds.

Axiom of Choice. Every system of sets has a choice function. [See Hungerford's Section 0.7 for details.]

As Hrbacek and Jech state (see page 225): "...the well-known concepts of real analysis...can be defined in set theory and their similar assertion can be made about any other branch of contemporary mathematics (except category theory [which Hungerford covers in his Section I.7]). Fundamental objects of topology, algebra, or functional analysis (say, topological spaces, vector spaces, groups, rings, Banach spaces) are customarily defined to be sets of a specific kind. Topologic, algebraic, and analytic properties of these objects are then derived from various properties of sets, which can be themselves in their turn obtained as consequences of the axioms of ZFC." Here's an image of Zermelo from the MacTutor History of Mathematics website:

Note. Ernst Zermelo (July 27, 1871 – May 21, 1953) published his axiomatic system in 1908 (for references and more details, see my online notes on Dr. Bob's Axiom of Choice Centennial Lecture). As a side note, Zermelo was an honorary chair at Freiberg University of Mining and Technology in Germany, a position he resigned from in 1935 because of the rise of the Nazis, but he was reinstated in 1946 and he died in Freiberg, Germany (see MacTutor Biography of Zermelo).

Definition. Two sets A and B are *equipollent* if there exists a bijective map from A to B, in which case we denote this as $A \sim B$.

Theorem 0.8.1. Equipollence is an equivalence relation on the class $\mathcal{S}$ of all sets.

Proof. A straighforward exercise.

Definition. If set A is equipollent to a set $I_n = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or to the set $i_0 = \varnothing$ then set A is *finite*. Otherwise set A is *infinite*.

Definition 0.8.2. The *cardinal number* (or *cardinality*) of a set A, denoted $|A|$, is the equivalence class of A under the equivalence relation of equipollence. $|A|$ is an infinite or finite cardinal according as to whether A is an infinite or finite set.

Example/Definition. The cardinal number of the set $\mathbb{N}$ of natural numbers is denoted $\aleph_0$ ("aleph naught"). A set A of cardinality $\aleph_0$ is *denumerable* (or *countable*). The sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are denumerable, but sets $\mathbb{R}$ and $\mathbb{C}$ are not denumerable.

Definition 0.8.3. Let α and β be cardinal numbers. The *sum* $\alpha+\beta$ is the cardinal number $|A \cup B|$, where A and B are disjoint sets such that $|A| = \alpha$ and $|B| = \beta$. The *product* $\alpha\beta$ is the cardinal number $|A \times B|$.

Note. In Exercise 0.8.5, some of the behavior of cardinal numbers is established: For cardinal numbers α, β, γ we have:

(a) $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$;

(b) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;

(c) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ and $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$;

(d) $\alpha + 0 = \alpha$ and $\alpha 1 = \alpha$;

(e) If $\alpha \neq 0$ then there is no β such that $\alpha + \beta = 0$ and if $\alpha \neq 1$ then there is no such β such that $\alpha\beta = 1$. Therefore subtraction and division of cardinal numbers cannot be defined.

Definition 0.8.4. Let α, β be cardinal numbers and A, B sets such that $|A| = \alpha$, $|B| = \beta$. α is *less than or equal to* β, denoted $\alpha \leq \beta$ or $\beta \geq \alpha$, if A is equipollent with a subset of β (that is, there is an injective map $A \to B$). α is *strictly less than* β, denoted $\alpha < \beta$ or $\beta > \alpha$, if $\alpha \leq \beta$ and $\alpha \neq \beta$.

Note. The following result, due to Georg Cantor (1845–1918) one of the founders of modern set theory, shows that there is no largest cardinal number with respect to $<$.

Theorem 0.8.5. If A is a set and $\mathcal{P}(A)$ is the power set, then $|A| < |\mathcal{P}(A)|$.

Note. Theorem 0.8.5 is commonly explained as imagining a town with a barber. This barber's job is to cut the hair of all those in the town who do not cut their own hair. The town is set A, "$a \in f(a)$" means that town member a cuts his/her own hair (and "$a \not\in f(a)$" means that a does not cut his/her own hair), and set B is the set of town members who get their hair cut by the barber. The contradiction is produced by asking: "Who cuts the barber's hair?" (Though, more traditionally, this story is used to explain Russell's Paradox, which implies that there can be no set of all sets.)

Note. For a finite set A with $|A| = n$, it is easy to show that $|\mathcal{P}(A)| = 2^n$ (by induction, say). Based on this fact, it is a common notation when $|A| = \alpha$ to denote $|\mathcal{P}(A)| = 2^\alpha$. From Theorem 0.8.5, we have $\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$. The real numbers, $\mathbb{R}$, are often called "the continuum" and the cardinality of the continuum is denoted $|\mathbb{R}| = c$. It can be shown that $c = |\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$ (see Section 6.2 of Hrbacek and Jech; this result required a rigorous definition of $\mathbb{R}$). Since $\aleph_0 < c$ it is natural to ask if there is a cardinal number β such that $\aleph_0 < \beta < c$. This is equivalent to the existence of a set B where $\mathbb{N} \subset B \subset \mathbb{R}$ where $|\mathbb{N}| < |B| < |\mathbb{R}|$.

Unfortunately, under the standard axioms of set theory, the ZFC axioms, the claim of the existence of such a set is “undecidable” (a concept introduced by Kurt Gödel [1906–1978]). In 1900, David Hilbert (1862–1943) gave a list of 23 unsolved problems at the International Congress of Mathematics in Paris and the first problem on the list was the “continuum problem.” The Continuum Hypothesis states that there is no such set B and no such cardinal number β. In 1939, Kurt Gödel proved that the Continuum Hypothesis is *consistent* with the ZFC axioms of set theory (in *The Consistency of the Continuum-Hypothesis*, Princeton University Press (1940)). In 1963, Paul Cohen (1934–2007) proved that the Continuum Hypothesis is *independent* of the ZFC axioms (in “The Independence of the Continuum Hypothesis,” *Proceedings of the National Academy of Sciences of the United States of America* **50**(6): 1144–1148 and “The Independence of the Continuum Hypothesis II,” ibid, **51**(1), 105–110). Cohen was awarded the Fields Medal in 1966 for his proof. This is why the Continuum Hypothesis is said to be undecidable within ZFC set theory. Under the assumption of the Continuum Hypothesis, we denote $\aleph_1 = 2^{\aleph_0} = c$ (strictly speaking, $\aleph_1$ has a definition based on the study of ordinal numbers, but we omit these details).

Note. We can muddle through the vast majority of Hungerford’s book with the set theory given above. However, some of the material in Hungerford requires the following results (namely, Theorem II.1.2, Theorem IV.2.6, Lemma V.3.5, Theorem V.3.6, and Theorem VI.1.9).

Note. Hrbacek and Jech call the following result the Cantor-Bernstein Theorem (see Section 4.1). We have little set theoretic background and our proof is somewhat unconventional.

Theorem 0.8.6. Schroeder-Bernstein Theorem.
If A and B are sets such that $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Theorem 0.8.7. The class of all cardinal numbers is linearly ordered by $\leq$. If α and β are cardinal numbers, then exactly on of the following is true:

$$\alpha < \beta; \;\; \alpha = \beta; \;\; \beta < \alpha.$$

The is called the *Law of Trichotomy.*

Theorem 0.8.8. Every infinite set has a denumerable subset. In particular $\aleph_0 \leq \alpha$ for every infinite cardinal number α.

Lemma 0.8.9. If A is an infinite set and F is a finite set then $|A \cup F| = |A|$. In particular, $\alpha + n = \alpha$ for every infinite cardinal number α and every finite cardinal number n.

Theorem 0.8.10. If α and β are cardinal numbers such that $\beta \leq \alpha$ and α is infinite, then $\alpha + \beta = \alpha$.

Note. The following two theorems are quoted in the proofs of Theorem II.1.2 and Theorem V.3.6.

Theorem 0.8.11. If α and β are cardinal numbers such that $0 \neq \beta \leq \alpha$ and α is infinite, then $\alpha\beta = \alpha$; in particular, $\alpha\aleph_0 = \alpha$ and if β is finite then $\aleph_0\beta = \aleph_0$.

Theorem 0.8.12. Let A be a set and for each integer $n \geq 1$ let $A^n = A \times A \times \cdots \times A$ (n factors).

(i) If A is finite, then $|A^n| = |A|^n$, and if A is infinite then $|A^n| = |A|$.

(ii) $|\cup_{n\in\mathbb{N}} A^n| = \aleph_0|A|$.

Note. The following result is used in the proof of Theorem IV.2.6.

Corollary 0.8.13. If A is an infinite set and $F(A)$ is the set of all finite subsets of A, then $|F(A)| = |A|$.

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