# **Supplement:** The Fundamental Theorem of Algebra—History

**Note.** There a number of ways to state the Fundamental Theorem of Algebra:

1. Every polynomial with complex coefficients has a complex root.
2. Every polynomial of degree $n$ with complex coefficients has $n$ complex roots counting multiplicity.
3. Every polynomial of degree $n$ with complex coefficients can be written as a product of linear terms (using complex roots).
4. The complex field is algebraically closed.

Historically, *finding* the roots of a polynomial has been the motivation for both classical and modern algebra.


Al-Khwarizmi (790–850) and Fibonacci (1170–1250)

(From MacTutor History of Mathematics)

**Note.** The Babylonians (1900 to 1600 BCE) had some knowledge of the quadratic equation and could solve the equation $x^2 + (2/3)x = 35/60$ (see page 1 Israel Kleiner's *A History of Abstract Algebra*, Birkhäuser: 2007). We would then expect that they could solve $x^2 + ax = b$ for $a > 0$ and $b > 0$. The technique would be to "complete the square" (suggestive geometric terminology, eh!). Of course, the Babylonians had no concept of what we call the quadratic equation:

$$xx^2 + bx + c = 0 \Longrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For more details on the development of ancient mathematics, including Euclid's presentations on number theory and the introduction of the "Arabic numerals" by Al-Khwarizmi and Fibonacci, see my handout for senior level Introduction to Modern Algebra (MATH 4127/5127) "A Student's Question: Why the Hell am I in this Class?":

`http://faculty.etsu.edu/gardnerr/4127/notes/Why-am-I-here.pdf`


Tartaglia (1500–1557) and Cardano (1501–1576)

(From MacTutor History of Mathematics)

**Note.** Around 1530, Niccolò Tartaglia discovered a formula for the roots of a third degree polynomial. Gerolamo Cardano published the formula in *Ars Magna* in 1545 (leading to a "battle" between Tartaglia and Cardano). In 1540, Ludovico Ferrari found a formula for the roots of a fourth degree polynomial. The cubic equation gives the roots of $ax^3 + bx^2 + cx + d = 0$ as:

$$x_1 = -\frac{b}{3a} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left(2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d^2)^2 - 4(b^2 - 3ac)^3}\right)}$$

$$-\frac{1}{3a}\sqrt[3]{\frac{1}{2}\left(2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d^2)^2 - 4(b^2 - 3ac)^3}\right)}$$

$$x_2 = -\frac{b}{3a} + \frac{1+\sqrt{-3}}{6a}\sqrt[3]{\frac{1}{2}\left(2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d^2)^2 - 4(b^2 - 3ac)^3}\right)}$$

$$+\frac{1-\sqrt{-3}}{6a}\sqrt[3]{\frac{1}{2}\left(2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d^2)^2 - 4(b^2 - 3ac)^3}\right)}$$

$$x_3 = -\frac{b}{3a} + \frac{1-\sqrt{-3}}{6a}\sqrt[3]{\frac{1}{2}\left(2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d^2)^2 - 4(b^2 - 3ac)^3}\right)}$$

$$+\frac{1+\sqrt{-3}}{6a}\sqrt[3]{\frac{1}{2}\left(2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d^2)^2 - 4(b^2 - 3ac)^3}\right)}$$

 

Abel (1802–1829) and Galois (1811–1832)

(From MacTutor History of Mathematics)

**Note.** The rapid succession of the discovery of the cubic and quartic equations lead many to think that a general (algebraic) formula for the roots of an $n$th degree polynomial was on the horizon. However, Niels Henrik Abel's 1821 proof of the unsolvability of the quintic and later work of Evaristé Galois, we now know that there is no such *algebraic* formula.

Albert Girard (1595–1632), Gottfried Leibniz (1646–1716), Leonhard Euler (1707–1783), Jean d'Alembert (1717–1783) (From MacTutor)

**Note.** The first person to clearly claim that an $n$ degree polynomial equation has $n$ solutions was Albert Girard (1595–1632) in 1629 in his *L'invention Nouvelle en l'Algèbre*. However, he did not understand the nature of complex numbers and this was to have implications for future explorations of the problem. In fact, Gottfried Wilhelm von Leibniz (1646–1716) claimed to prove that the Fundamental Theorem of Algebra was false, as shown by considering $x^4 + t^4$ which he claimed could not be written as a product of two real quadratic factors. His error was based, again, on a misunderstanding of complex numbers. In 1742, Leonhard Euler (1707–1783) showed that Leibniz's example was not correct. In 1746, Jean Le Rond D'Almbert (1717–1783) made the first serious attempt at a proof of the Fundamental Theorem of Algebra, but his proof had several weaknesses. This note is based on:

`http://www-history.mcs.st-and.ac.uk/HistTopics/`

`Fund_theorem_of_algebra.html`

**Note.** Leonhard Euler proved that every real polynomial of degree $n$, where $n \leq 6$, has exactly $n$ complex roots. In 1749, Euler attempted a proof for a general $n$th degree polynomial, but his proof was a bit sketchy. In 1772, Joseph-Louis Lagrange raised objections to Euler's proof. Pierre-Simon Laplace (1749–1827), in 1795, tried to prove the FTA using a completely different approach using the discriminant of a polynomial. His proof was very elegant and its only 'problem' was that again the existence of roots was assumed. (See the reference for the previous note.)

Carl Friederich Gauss (1777–1855)
(From MacTutor History of Mathematics)

**Note.** Quoting from Morris Kline's *Mathematical Thought from Ancient to Modern Times*, Oxford University Press (1972), Volume 2 (page 598): "The first substantial proof of the fundamental theorem, though not rigorous by modern standards, was given by Gauss in his doctoral thesis of 1799 at Hemlstädt" (see *Werke*, Königliche Gellschaft der Wissenscheften zu Göttingen, 1876, **3** (1–30)). "He criticized the work of d'Alembert, Euler, and Lagrange and then gave his own proof. Gauss's method was not to calculate a root but to demonstrate its existence. . . . Gauss gave three more proofs of the theorem."

**Note.** Gauss's first proof, given in his dissertation, was a geometric proof which depended on the intersection of two curves which were based on the polynomial. In his second proof, he abandoned the geometric argument, but gave an argument still not rigorous based on the ideas of the time [*Werke*, **3**, 33–56]. The third proof was based on Cauchy's Theorem and, hence, on the then-developing theory of complex functions (see "Gauss's Third Proof of the Fundamental Theorem of Algebra," *American Mathematical Society, Bulletin*, **1** (1895), 205-209). This was originally published in 1816 in *Comm. Soc. Gott*, **3** (also in *Werkes*, **3**, 59–64). The fourth proof is similar to his first proof and appears in *Werke*, **3**, 73–102 (originally published in *Abhand. der Ges. der Wiss. zu Gött.*, **4**, 1848/50). Gauss's proofs were not entirely general, in that the first three proofs assumed that the coefficients of the polynomial were real. Gauss's fourth proof covered polynomials with complex coefficients. Gauss's work was ground-breaking in that he demonstrated the existence of the roots of a polynomial without actually *calculating* the roots [Kline, pages 598 and 599].

Joseph Liouville (1809–1882)

(From MacTutor History of Mathematics)

**Note.** To date, the easiest proof is based on Louisville's Theorem (which, like Gauss's third proof is, in turn, based on Cauchy's Theorem). Louisville's Theorem appears in 1847 in "*Le cons sur les fonctions doublement périodiques*," *Journal für Mathematik Bb.*, **88**(4), 277–310.

**Note.** In your graduate career, you have several opportunities to see a proof of the Fundamental Theorem of Algebra. Here are some of them:

**1.** In Complex Analysis [MATH 5510/5520] where Liouville's Theorem is used to give a very brief proof. See

`http://faculty.etsu.edu/gardnerr/5510/notes/IV-3.pdf`

(Theorems IV.3.4 and IV.3.5). You are likely to see the same proof in our Complex Variables class [MATH 4337/5337]. In fact, this is the proof which Fraleigh presents in his *A First Course In Abstract Algebra*, 7th Edition:

`http://faculty.etsu.edu/gardnerr/4127/notes/VI-31.pdf`

(see Theorem 31.18).

**2.** In Complex Analysis [MATH 5510/5520] *again* where Rouche's Theorem (based on the argument principle) is used:

`http://faculty.etsu.edu/gardnerr/5510/notes/V-3.pdf`

(see Theorem V.3.8 and page 4).

**3.** In Introduction to Topology [MATH 4357/5357] where path homotopies and fundamental groups of a surface are used:

`http://faculty.etsu.edu/gardnerr/5210/notes/Munkres-56.pdf`.

**4.** In Modern Algebra 2 [MATH 5420] where a mostly algebraic proof is given, but two assumptions based on analysis are made: **(A)** every positive real number has a real positive square root, and **(B)** every polynomial in $\mathbb{R}[x]$ of odd degree has a root in $\mathbb{R}$. Both of these assumptions are based on the definition of $\mathbb{R}$ and the Axiom of Completeness. See:

`http://faculty.etsu.edu/gardnerr/5410/notes/V-3-A.pdf`

**Note.** There are no *purely* algebraic proofs of the Fundamental Theorem of Algebra [*A History of Abstract Algebra*, Israel Kleiner, Birkhäuser (2007), page 12]. There are proofs which are mostly algebraic, but which borrow result(s) from analysis (such as the proof presented by Hungerford). However, if we are going to use a result from analysis, the easiest approach is to use Liouville's Theorem from complex analysis. This leads us to a philosophical question concerning the legitimacy of the title "Fundamental Theorem of *Algebra*" for this result! If seems more appropriate to refer to it as "Liouville's Corollary"! Polynomials with complex coefficients are best considered as special analytic functions (an analytic function is one with a power series representation) and are best treated in the realm of complex analysis. Your humble instructor therefore argues that the Fundamental Theorem of Algebra is actually a result of some moderate interest in the theory of analytic complex functions. After all, *algebra* in the modern sense does not deal so much with polynomials (though this is a component of modern algebra), but instead deals with the theory of groups, rings, and fields!

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