Section I.4. Cosets and Counting

Note. In this section, we generalize the idea of congruence modulo m on $\mathbb{Z}$ to a more general setting. This is the same approach taken in an undergraduate class, but we will deal in a more hands-on way with the equivalence relation here.

Definition I.4.1. Let H be a subgroup of group G and $a, b \in G$. a is *right congruent* to b modulo H, denoted $a \equiv_r b \pmod{H}$ if $ab^{-1} \in H$. a is *left congruent* to b modulo H, denoted $a \equiv_\ell b \pmod{H}$, if $a^{-1}b \in H$.

Note. We use left and right congruent to define left and right cosets. As in undergraduate algebra, we'll use the cosets to prove Lagrange's Theorem and, under "appropriate conditions" make a group out of the cosets.

Theorem I.4.2. Let H be a subgroup of a group G.

(i) Right and left congruence modulo H are each equivalence relations on G.

(ii) The equivalence class of $a \in G$ under right (and left) congruence modulo H is the set $Ha = \{ha \mid h \in H\}$ (and $aH = \{ah \mid h \in H\}$ for left congruence).

(iii) $|Ha| = |H| = |aH|$ for all $a \in G$.

The set Ha is a *right coset* of H in G and aH is a *left coset* of H in G.

Note. We see from the proof that $|Ha| = |aH| = |H|$, even if H is infinite since the existence of a bijection is established.

Corollary I.4.3. Let H be a subgroup of group G.

(i) G is the union of the right (and left) cosets of H in G.

(ii) Two right (or two left) cosets of H in G are either disjoint or equal.

(iii) For $a, b \in G$, we have that $Ha = Hb$ if and only if $ab^{-1} \in H$, and $aH = bH$ if and only if $a^{-1}b \in H$.

(iv) If $\mathcal{R}$ is the set of distinct right cosets of H in G and $\mathcal{L}$ is the set of distinct left cosets of H in G, then $|\mathcal{R}| = |\mathcal{L}|$.

Note. Parts (i) and (ii) imply that the right (and left) cosets of H in G partition G.

Note. In additive notation, we have $a \cong_r b \pmod{H}$ if and only if $a - b \in H$. The equivalence class of $a \in G$ is the right coset $H + a = \{h + a \mid h \in H\}$.

Definition I.4.4. Let H be a subgroup of a group G. The *index* of H in G, denoted $[G : H]$, is the cardinal number of the set of distinct right (or left) cosets of H in G.

Note. G and H may be infinite while $[G : H]$ is finite: $[\mathbb{Z} : \langle m \rangle] = m$. The extreme values of the index occur when $H = \{e\}$ and $H = G$: $[G : \{e\}] = |G|$ and $[G : G] = 1$.

Theorem I.4.5. If K, H, G are groups with $K < H < G$, then $[G : K] = [G : H][H : K]$. If any two of these indices are finite, then so is the third.

Note. The proof of the well-known Lagrange's Theorem is now easy.

Corollary I.4.6. Lagrange's Theorem.

If H is a subgroup of a group G, then $|G| = [G : H]|H|$. In particular, if G is finite then the order $|a|$ of $a \in G$ divides $|G|$; $|H|$ divides $|G|$.

Note. The converse of the "in particular" comment in Lagrange's Theorem does not hold. For example, the alternating group A_4 of order 12 (defined in Section I.6) does not have a subgroup of order 6; this is to be shown in Exercise I.6.8. So it is natural to ask: "For a given divisor d of the order of a finite group G, under what conditions does G have a subgroup of order d?" This is partially addressed in Section II.5. The Sylow Theorems; see Cauchy's Theorem (Theorem II.5.2) and the First Sylow Theorem (Theorem II.5.7).

Note. For group G and H, K subsets of G, we denote the set $\{hk \mid h \in H, k \in K\}$ as HK. If H and K are subgroups of G, then HK may or may not (see Exercise I.4.7) be a subgroup of G. Now for some "counting" results.

Theorem I.4.7. Let H and K be finite subgroups of a group G. Then $|HK| = |H||K|/|H \cap K|$.

Proposition I.4.8. If H and K are subgroups of a group G, then $[H : H \cap K] \leq [G : K]$. If $[G : K]$ is finite, then $[H : H \cap K] = [G : K]$ if and only if $G = KH$.

Proposition I.4.9. Let H and K be subgroups of finite index of group G. Then $[G : H \cap K]$ is finite and $[G : H \cap K] \leq [G : H][G : K]$. Furthermore, $[G : H \cap K] = [G : H][G : K]$ if and only if $G = HK$.

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