Section I.5. Normality, Quotient Groups, and Homomorphisms

Note. In this one section, we introduce and relate several ideas. You probably spent a significant part of your undergraduate algebra class on these topics. As you know, the study of quotient groups (or "factor groups" as Fraleigh calls them) are fundamental to the study of group theory.

Theorem I.5.1. If N is a subgroup of group G, then the following conditions are equivalent.

(i) Left and right congruence modulo N coincide (that is, define the same equivalence relation on G);

(ii) Every left coset of N in G is a right coset of N in G;

(iii) $aN = Na$ for all $a \in G$;

(iv) For all $a \in G$, $aNa^{-1} \subset N$ where $aNa^{-1} = \{ana^{-1} \mid n \in N\}$;

(v) For all $a \in G$, $aNa^{-1} = N$.

Definition I.5.2. A subgroup N of a group G which satisfies the equivalent conditions of Theorem 5.1 is said to be *normal* in G (or a *normal subgroup* of G), denoted $N \lhd G$.

Note I.5.A. Trivially, every subgroup of an abelian group is normal (since every left coset is a right coset, say). Any subgroup N of index 2 in group G is a normal subgroup by Exercise I.5.1. The intersection of any collection of normal subgroups of group G is itself a normal subgroup by Exercise I.5.2.

Note I.5.B. It may be the case that for group G, we have subgroups of G satisfying $N \triangleleft M$ and $M \triangleleft G$ but N is not a normal subgroup of G (see Exercise I.5.10). However if $N \triangleleft G$ and $N < M < G$ then $N \triangleleft M$. These observations (along with Exercise I.5.1 mentioned above) will play a role when we explore series of groups in Section II.8.

Theorem I.5.3. Let K and N be subgroups of a group G with N normal in G. Then

(i) $N \cap K \triangleleft K$;

(ii) $N \triangleleft N \vee K$;

(iii) $NK = N \vee K = KN$;

(iv) If $K \triangleleft G$ and $K \cap N = \{e\}$, then $nk = kn$ for all $k \in K$ and $n \in N$.

Note. The following result is a big one! It says that if N is a normal subgroup of G, then we can form a group out of the cosets of N by defining a binary operation on the cosets using representatives of the cosets. In fact, this can be done only when N is a normal subgroup, as seen in undergraduate algebra. You'll recall that the group of cosets is called a "factor group" or "quotient group." Quotient groups are at the backbone of modern algebra!

Theorem I.5.4. If N is a normal subgroup of a group G and G/N is the set of all (left) cosets of N in G, then G/N is a group of order $[G : N]$ under the binary operation given by $(aN)(bN) = (ab)N$.

Note I.5.C. By the definition of of the binary operation in G/N, we see that the identity element is $eN = N$, and the inverse of aN is $a^{-1}N$.

Definition. If N is a normal subgroup of G, then the group G/N of Theorem 5.4 is the *quotient group* (or *factor group*) of G by N.

Note. The relationship between quotient groups and normal subgroups is a little deeper than Theorem I.5.4 implies. From Fraleigh, we have:

> **Theorem 14.4 (Fraleigh).** Let H be a subgroup of a group G. Then left coset multiplication is well defined by $(aH)(bH) = (ab)H$ if and only if H is a normal subgroup of G.

This result gives the real importance of normal subgroups.

Note I.5.D. Recall (see Section I.4) that the equivalence classes of $\mathbb{Z}$ modulo m are the cosets of $\langle m\rangle$ in $\mathbb{Z}$ ($\mathbb{Z}$ is an additive group, so the cosets are of the form $k + \langle m\rangle$). Since $\mathbb{Z}$ is an abelian group, subgroup $\langle m\rangle$ is a normal subgroup of $\mathbb{Z}$ and so the quotient group $\mathbb{Z}/\langle m\rangle$ exists. In fact, $\mathbb{Z}_m = \mathbb{Z}/\langle m\rangle$.

Note. Fraleigh introduces quotient groups by first considering the kernel of a homomorphism and later considering normal subgroups. Hungerford is covering the same material but in the opposite order.

Theorem I.5.5. If $f : G \to H$ is a homomorphism of groups, then the kernel of f is a normal subgroup of G. Conversely, if N is a normal subgroup of G, then the map $\pi : G \to G/N$ given by $\pi(a) = aN$ is an epimorphism (that is, an onto homomorphism) with kernel N.

Definition. The map $\pi : G \to G/N$ of Theorem I.5.5 defined as $\pi(a) = aN$ is the *canonical epimorphism*. (Fraleigh called this the "canonical homomorphism.")

Theorem I.5.6. (Similar to Fraleigh's "Fundamental Homomorphism Theorem;" Theorem 14.11) If $f : G \to H$ is a homomorphism and N is a normal subgroup of G contained in the kernel of f, then there is a unique homomorphism $\overline{f} : G/N \to H$ such that $\overline{f}(aN) = f(a)$ for all $a \in G$. Also, $\mathrm{Im}(f) = \mathrm{Im}(\overline{f})$ and $\mathrm{Ker}(\overline{f}) = \mathrm{Ker}(f)/N$. $\overline{f}$ is an isomorphism if and only if f is an epimorphism and $N = \mathrm{Ker}(f)$.

Note. On page 4, Hungerford defines the idea of a "commutative diagram." Basically, a diagram (represented by a directed graph where the arcs represent functions and the vertices represent groups) is commutative if following one sequence of arcs from a beginning group to a final group yields a composition of functions which is equal to the resulting composition of any other such sequence of arcs. Theorem I.5.6, in part, claims that the following diagram commutes:

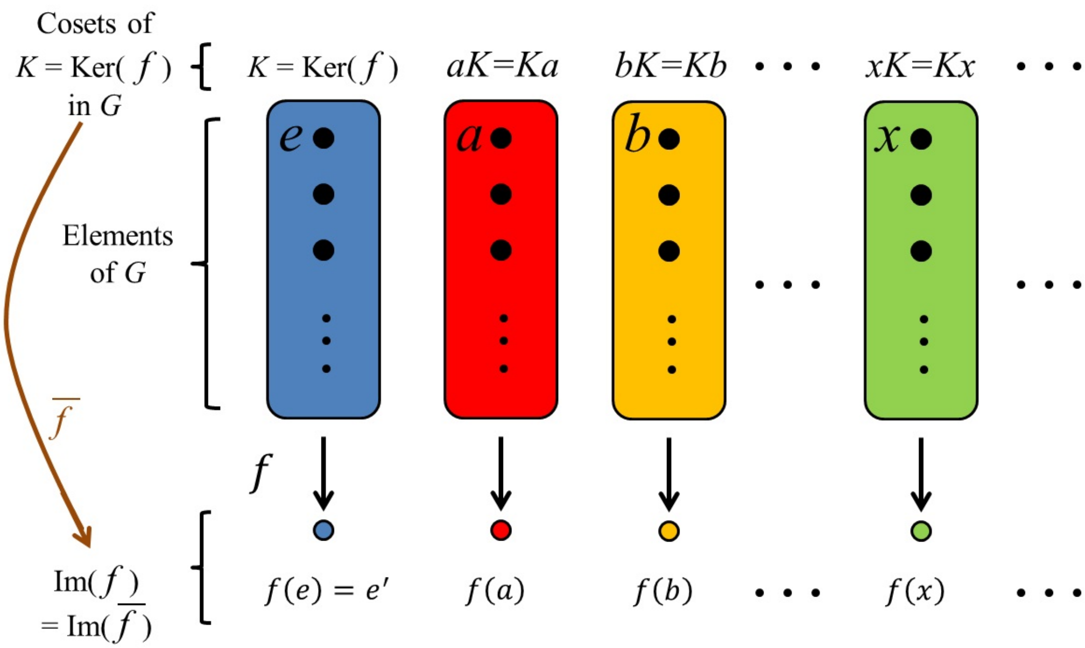
$$
\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\
G/N \arrow[ur, "\overline{f}"'] &
\end{tikzcd}
$$

Note. We now state and prove the three "Isomorphism Theorems" which relate groups and quotient groups. This topic is covered in Fraleigh's Part VII, Advanced Group Theory, Section 34: Isomorphism Theorems.

Corollary I.5.7. First Isomorphism Theorem.
If $f : G \to H$ is a homomorphism of groups, then f induces an isomorphism of $G/\mathrm{Ker}(f)$ with $\mathrm{Im}(f)$.

Proof. Since $f : G \to \mathrm{Im}(f)$ is an onto homomorphism (i.e., an epimorphism), then Theorem I.5.6 applies to give isomorphism $\overline{f} : G/N \to \mathrm{Im}(f) = \mathrm{Im}(\overline{f})$ where $N = \mathrm{Ker}(f)$. ∎

Note. The First Isomorphism Theorem (and its close relatives, such as Theorem I.5.6) is fundamental in modern algebra. It relates quotient groups ("factor groups"), which have cosets as elements, to a group homomorphism and its kernel. The following diagram shows the elements of G, the cosets of $\mathrm{Ker}(f)$, and the mappings f and $\overline{f}$. You may recognize this image since it is very similar to the logo of the ETSU Abstract Algebra Club.



Corollary I.5.8. If $f : G \to H$ is a homomorphism of groups, $N \triangleleft G$, $M \triangleleft H$, and $f(N) < M$, then f induces a homomorphism $\overline{f} : G/N \to H/M$, given by $aN \mapsto f(a)M$. f is an isomorphism if and only if $\mathrm{Im}(f) \vee M = H$ and $f^{-1}(M) \subset N$. In particular if f is an epimorphism such that $f(N) = M$ and $\mathrm{Ker}(f) \subset N$, then $\overline{f}$ is an isomorphism.

Corollary I.5.9. Second Isomorphism Theorem.

If K and N are subgroups of a group G, with N normal in G, then $K/(N \cap K) \cong NK/N$.

Example. (Fraleigh's Exercise 34.4.) For $G = \mathbb{Z}_{36}$, let $K = \langle \overline{6} \rangle$ and $N = \langle \overline{9} \rangle$.

(a) List the elements in KN (which should be written $K + N$ since these are additive groups) and $K \cap N$.

Solution. We have $K = \langle \overline{6} \rangle = \{\overline{0}, \overline{6}, \overline{12}, \overline{18}, \overline{24}, \overline{30}\}$ and $N = \langle \overline{9} \rangle = \{\overline{0}, \overline{9}, \overline{18}, \overline{27}\}$. So $K + N = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}, \overline{30}, \overline{33}\}$ and $K \cap N = \{\overline{0}, \overline{18}\}$.

(b) List the cosets of $(KN)/N = (K + N)/N$.

Solution. The cosets are $N + \overline{0} = N + \overline{9} = N + \overline{18} = N + \overline{27} = \{\overline{0}, \overline{9}, \overline{18}, \overline{27}\}$, $N + \overline{3} = N + \overline{12} = N + \overline{21} = N + \overline{30} = \{\overline{3}, \overline{12}, \overline{21}, \overline{30}\}$, and $N + \overline{6} = N + \overline{15} = N + \overline{24} = N + \overline{33} = \{\overline{6}, \overline{15}, \overline{24}, \overline{33}\}$.

(c) List the cosets of $K/(K \cap N)$.

Solution. The cosets are $K \cap N + \overline{0} = K \cap N + \overline{18} = \{\overline{0}, \overline{18}\}$, $K \cap N + \overline{6} = K \cap N + \overline{24} = \{\overline{6}, \overline{24}\}$, and $K \cap N + \overline{12} = K \cap N + \overline{30} = \{\overline{12}, \overline{30}\}$.

(d) Give the correspondence between $(KN)/N = (K + N)/N$ and $K/(K \cap N)$ described in the proof of Corollary I.5.9.

Solution. In the notation of Corollary I.5.9, we have $\overline{f} : K/(K \cap N) \to (K+N)/N$ where $\overline{f}(k+(K\cap N)) = (k+n)+N$, so $\overline{f}(\overline{0}+K\cap N) = (\overline{0}+\overline{0})+N = (\overline{0} + \overline{9}) + N = (\overline{0} + \overline{18}) + N = (\overline{0} + \overline{27}) + N$, $\overline{f}(\overline{6} + K \cap N) = (\overline{6} + \overline{0}) + N = (\overline{6}+\overline{9})+N = (\overline{6}+\overline{18})+N = (\overline{6}+\overline{27})+N$, and $\overline{f}(\overline{12}+K\cap N) = (\overline{12}+\overline{0})+N = (\overline{12}+\overline{9})+N = (\overline{12}+\overline{18})+N = (\overline{12}+\overline{27})+N$. So $\overline{f}$ gives $\{\overline{0}, \overline{18}\} \to \{\overline{0}, \overline{9}, \overline{18}, \overline{27}\}$, $\{\overline{6}, \overline{24}\} \to \{\overline{6}, \overline{15}, \overline{24}, \overline{33}\}$, and $\{\overline{12}, \overline{30}\} \to \{\overline{3}, \overline{12}, \overline{21}, \overline{30}\}$.

Corollary I.5.10. Third Isomorphism Theorem.

If H and K are normal subgroups of a group G such that $K < H$, then H/K is a normal subgroup of G/K and $(G/K)/(H/K) \cong G/H$.

Example. (Fraleigh's Exercise 34.5.) Let $G = \mathbb{Z}_{24}$, $H = \langle \overline{4} \rangle$, and $K = \langle \overline{8} \rangle$.

(a) List the cosets in G/H.

Solution. We have $H = \langle \overline{4} \rangle = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}\}$ and $K = \langle \overline{8} \rangle = \{\overline{0}, \overline{8}, \overline{16}\}$. The cosets are $H + \overline{0} = H + \overline{4} = H + \overline{8} = H + \overline{12} = H + \overline{16} = H + \overline{20} = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}\}$, $H + \overline{1} = H + \overline{5} = H + \overline{9} = H + \overline{13} = H + \overline{17} = H + \overline{21} = \{\overline{1}, \overline{5}, \overline{9}, \overline{13}, \overline{17}, \overline{21}\}$, $H + \overline{2} = H + \overline{6} = H + \overline{10} = H + \overline{14} = H + \overline{18} = H + \overline{22} = \{\overline{2}, \overline{6}, \overline{10}, \overline{14}, \overline{18}, \overline{22}\}$, and $H + \overline{3} = H + \overline{7} = H + \overline{11} = H + \overline{15} = H + \overline{19} = H + \overline{23} = \{\overline{3}, \overline{7}, \overline{11}, \overline{15}, \overline{19}, \overline{23}\}$.

(b) List the cosets in G/K.

Solution. The cosets are $K + \overline{0} = K + \overline{8} = K + \overline{16} = \{\overline{0}, \overline{8}, \overline{16}\}$, $K + \overline{1} = K + \overline{9} = K + \overline{17} = \{\overline{1}, \overline{9}, \overline{17}\}$, $K + \overline{2} = K + \overline{10} = K + \overline{18} = \{\overline{2}, \overline{10}, \overline{18}\}$, $K + \overline{3} = K + \overline{11} = K + \overline{19} = \{\overline{3}, \overline{11}, \overline{19}\}$, $K + \overline{4} = K + \overline{12} = K + \overline{20} = \{\overline{4}, \overline{12}, \overline{20}\}$, $K + \overline{5} = K + \overline{13} = K + \overline{21} = \{\overline{5}, \overline{13}, \overline{21}\}$, $K + \overline{6} = K + \overline{14} = K + \overline{22} = \{\overline{6}, \overline{14}, \overline{22}\}$, and $K + \overline{7} = K + \overline{15} = K + \overline{23} = \{\overline{7}, \overline{15}, \overline{23}\}$.

(c) List the cosets in H/K.

Solution. The cosets are $K + \overline{0} = K + \overline{8} = K + \overline{16} = \{\overline{0}, \overline{8}, \overline{16}\}$, and $K + \overline{4} = K + \overline{12} = K + \overline{20} = \{\overline{4}, \overline{12}, \overline{20}\}$.

(d) List the cosets in $(G/K)/(H/K)$.

Solution. The cosets are of the form (element of G/K) + (H/K), since H/K

consists of two cosets, we have that the cosets are $(K+\overline{0})+(H/K)=(K+\overline{4})+(H/K)=\{\overline{0},\overline{8},\overline{16}\}+\{\{\overline{0},\overline{8},\overline{16}\},\{\overline{4},\overline{12},\overline{20}\}\}=\{\{\overline{0},\overline{8},\overline{16}\},\{\overline{4},\overline{12},\overline{20}\}\}$, $(K+\overline{1})+(H/K)=(K+\overline{5})+(H/K)=\{\overline{1},\overline{9},\overline{17}\}+\{\{\overline{0},\overline{8},\overline{16}\},\{\overline{4},\overline{12},\overline{20}\}\}=\{\{\overline{1},\overline{9},\overline{17}\},\{\overline{5},\overline{13},\overline{21}\}\}$, $(K+\overline{2})+(H/K)=(K+\overline{6})+(H/K)=\{\overline{2},\overline{10},\overline{18}\}+\{\{\overline{0},\overline{8},\overline{16}\},\{\overline{4},\overline{12},\overline{20}\}\}=\{\{\overline{2},\overline{10},\overline{18}\},\{\overline{6},\overline{14},\overline{22}\}\}$, and $(K+\overline{3})+(H/K)=(K+\overline{7})+(H/K)=\{\overline{3},\overline{11},\overline{19}\}+\{\{\overline{0},\overline{8},\overline{16}\},\{\overline{4},\overline{12},\overline{20}\}\}=\{\{\overline{3},\overline{11},\overline{19}\},\{\overline{7},\overline{15},\overline{23}\}\}$.

(e) Give the correspondence ("natural isomorphism") between G/H and $(G/K)/(H/K)$.

Solution. The natural isomorphism maps $H+\overline{i}$ to $(K+\overline{i})+H/K$ for $i=0,1,2,3$. So $\{\overline{0},\overline{4},\overline{8},\overline{12},\overline{16},\overline{20}\}\to\{\{\overline{0},\overline{8},\overline{16}\},\{\overline{4},\overline{12},\overline{20}\}\}$, $\{\overline{1},\overline{5},\overline{9},\overline{13},\overline{17},\overline{21}\}\to\{\{\overline{1},\overline{9},\overline{17}\},\{\overline{5},\overline{13},\overline{21}\}\}$, $\{\overline{2},\overline{6},\overline{10},\overline{14},\overline{18},\overline{22}\}\to\{\{\overline{2},\overline{10},\overline{18}\},\{\overline{6},\overline{14},\overline{22}\}\}$, and $\{\overline{3},\overline{7},\overline{11},\overline{15},\overline{19},\overline{23}\}\to\{\{\overline{3},\overline{11},\overline{19}\},\{\overline{7},\overline{15},\overline{23}\}\}$ (where $i=0,1,2,3$, respectively).

Theorem I.5.11. If $f:G\to H$ is an epimorphism of groups, then the assignment $K\mapsto f(K)$ defines a one-to-one correspondence between the sets $S_f(G)$ of all subgroups K of G which contain $\mathrm{Ker}(f)$ and the set $S(H)$ of all subgroups of H. Under this correspondence normal subgroups correspond to normal subgroups.

Corollary I.5.12. If N is a normal subgroup of a group G, then every subgroup of G/N is of the form K/N, where K is a subgroup of G that contains N. Furthermore, K/N is normal in G/N if and only if K is normal in G.

Note I.5.E. We give a final warning about quotient groups! It is possible for $G_1 \cong G_2$, $H_1 \triangleleft G_1$, $H_2 \triangleleft G_2$, and $H_1 \cong H_2$, BUT $G_1/H_1 \not\cong G_2/H_2$. Consider $G_1 = G_2 = \mathbb{Z}$, $H_1 = 2\mathbb{Z}$, and $H_2 = 3\mathbb{Z}$. Then $H_1 \cong H_2 \cong \mathbb{Z}$. However, $G_1/H_1 = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ and $G_2/H_2 = \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}_3$. So $G_1/H_1 = \mathbb{Z}_2 \not\cong \mathbb{Z}_3 = G_2/H_2$. The implication here is that quotient groups behave in some possibly unpredictable ways.

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