Section I.7. Categories: Products, Coproducts, and Free Objects

Note. Sections I.1 to I.6 contain topics which you encountered in undergraduate algebra. The remainder of Chapter I contains topics you probably did not encounter in your undergraduate sequence. Hungerford uses terminology from the area of "categories" in the rest of Chapter I and occasionally throughout the rest of the book. The last chapter of the book, Chapter X, is devoted to category theory (about 20 pages). By comparison, Dummit and Foote (*Abstract Algebra*, 3rd Edition, 2004) mentions category theory in the body of their book, but mostly restricts it to an appendix.

Note. Informally, a "category" is a class of mathematical objects (eg., the category of groups, the category of sets, etc.). Russell's Paradox shows that we cannot have a "*set* of all sets," however category theory allows a *category* of all sets (see *Categories for the Working Mathematician*, S. MacLane, Springer-Verlag, 1971). The idea is to collect into a category all similar mathematical objects and then to give a proof of some property in the setting of category theory—the result then applies to all categories and hence in different mathematical settings (eg., groups, sets, etc.).

Note. Quoting from Chapter X (page 464): "Many different mathematical topics may be interpreted in terms of categories so that the techniques and theorems of the theory of categories may be applied to these topics. ...Consequently it is frequently possible to provide a proof in a category setting, which has as special cases the previously known results from two different areas. This unification process provides a means of comprehending wider areas of mathematics as well as new topics whose fundamentals are expressible in categorical terms."

Definition I.7.1. A *category* is a class $\mathcal{C}$ of objects (denoted A, B, C) together with

(i) a class of disjoint sets, denoted $\hom(A, B)$, one for each pair of objects in $\mathcal{C}$ (an element f of $\hom(A, B)$ is called a *morphism* from A to B and is denoted $f : A \to B$); and

(ii) for each triple (A, B, C) of objects of $\mathcal{C}$ a function mapping

$$\hom(B, C) \times \hom(A, B) \to \hom(A, C)$$

(for morphisms $f : A \to B$, $g : B \to C$, this function is written $(g, f) \mapsto g \circ f$ and $g \circ f : A \to C$ is called the *composite* of f and g). All such functions are subject to the two axioms:

(I) Associativity. If $f : A \to B$, $g : B \to C$, $h : C \to D$ are morphisms of $\mathcal{C}$ then $h \circ (g \circ f) = (h \circ g) \circ f$.

(II) Identity. For each object B of $\mathcal{C}$ there exists a morphism $1_B : B \to B$ such that for any $f : A \to B$, $g : B \to C$

$$1_B \circ f = f \text{ and } g \circ 1_B = g.$$

Definition. In a category $\mathcal{C}$ a morphism $f : A \to B$ is called an *equivalence* if there is in $\mathcal{C}$ a morphism $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. If $f : A \to B$ is an equivalence then A and B are said to be *equivalent.*

Example. Let $\mathcal{S}$ be the category of all sets. For $A, B \in \mathcal{S}$, $\hom(A, B)$ is the set of all functions $f : A \to B$. Function composition is associative:

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x))) = (h \circ g)(f(x)) = ((h \circ g) \circ f)(x).$$

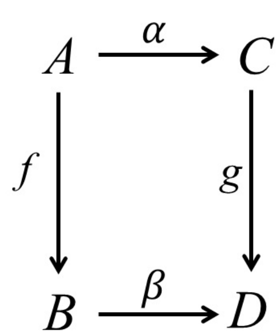
Clearly, for any B, 1_B is defined as $1_B(b) = b$ for all $b \in B$. Function f is an equivalence if f is invertible and f is invertible if and only if f is bijective by Equation (13) of Section 0.3.

Example. Let $\mathcal{G}$ be the category of all groups with $\hom(A, B)$ as the set of all group homomorphisms $f : A \to B$. Associativity and Identity are satisfied as in the previous example. By Theorem I.2.3(ii), a morphism is an equivalence if and only if it is an isomorphism. The category $\mathcal{A}$ of all abelian groups is defined similarly.

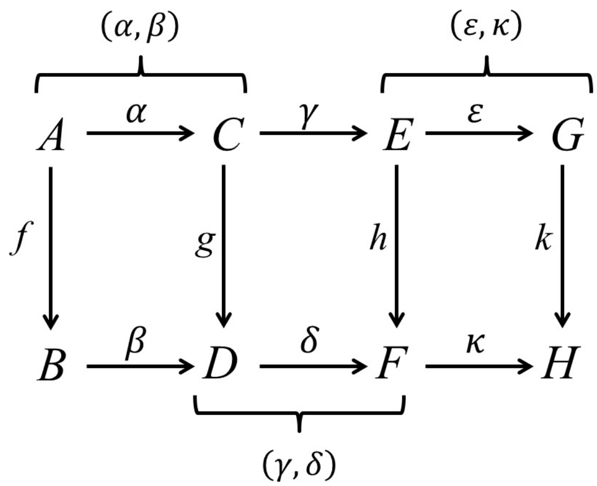
Note. The above two examples help illustrate the idea of equivalence. In $\mathcal{S}$, two sets are equivalent if and only if they are of the same cardinality. In $\mathcal{G}$, two groups are equivalent if and only if they are isomorphic.

Example I.7.A. To illustrate that a category need not consist of *all* sets or *all* groups, a multiplicative group G can be considered as a category with one element, G. Let $\hom(G, G)$ be the set of elements of G and $a \circ b = a * b$. Then $1_G = e$ and since every element of $\hom(G, G)$ has an inverse then every element is an equivalence. The multiplication of elements under the binary operation insures that the definition of category is satisfied.

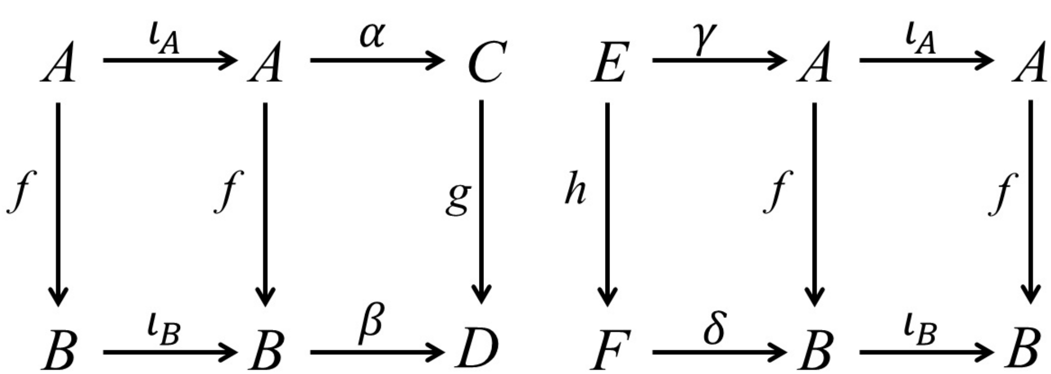
Example. Let $\mathcal{C}$ be a category and define the category $\mathcal{D}$ whose objects are all morphisms of $\mathcal{C}$. If $f : A \to B$ and $g : C \to D$ are morphisms of $\mathcal{C}$ then $\hom(f, g)$ consists of all pairs (α, β) of morphisms of $\mathcal{C}$ such that $\alpha : A \to C$, $\beta : B \to D$, and $\beta \circ f : A \to D$ is the same as $g \circ \alpha : A \to D$. That is, the following diagram is commutative:



When these conditions are satisfied, we consider $(\alpha, \beta) : f \to g$. For associativity, suppose $(\alpha, \beta) : f \to g$, $(\gamma, \delta) : g \to h$, and $(\varepsilon, \kappa) : h \to k$; suppose $f : A \to B$, $g : C \to D$, $h : E \to F$, and $k : G \to H$. So we have (based on the setting) $\alpha : A \to C$, $\beta : B \to D$, $\gamma : C \to E$, $\delta : D \to F$, $\varepsilon : E \to G$, and $\kappa : F \to H$; and $\beta \circ f = g \circ \alpha$, $\delta \circ g = h \circ \gamma$, $\kappa \circ h = k \circ \varepsilon$. Schematically:



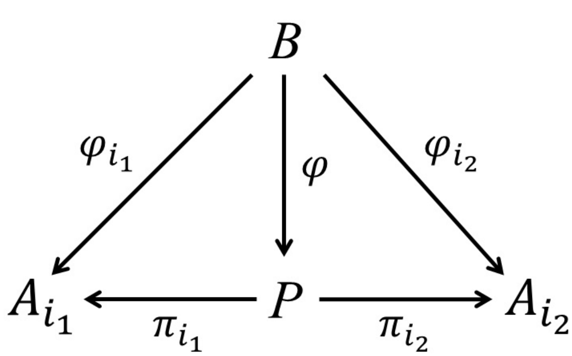
Now $(\gamma,\delta)\circ(\alpha,\beta) : f \to h$, where $f : A \to B$, $h : E \to F$, $\gamma\circ\alpha : A \to E$, $\delta\circ\beta : B \to F$, and we know that $h\circ(\gamma\circ\alpha) : A \to F$ is the same as $(\delta\circ\beta)\circ f : A \to F$. Also, $(\varepsilon\circ\kappa)\circ((\gamma,\delta)\circ(\alpha,\beta)) : f \to k$ where $(k\circ\varepsilon)\circ(\gamma\circ\alpha) : A \to H$ is the same as $(\kappa\circ\delta)\circ(\beta\circ f) : A \to H$. Similarly $((\varepsilon,\kappa)\circ(\gamma,\delta))\circ(\alpha,\beta) : f \to k$ and because function composition is associative $(\varepsilon,\kappa)\circ((\gamma,\delta)\circ(\alpha,\beta)) = ((\varepsilon,\kappa)\circ(\gamma,\delta))\circ(\alpha,\beta)$. For Identity, suppose $f, g, h \in \mathcal{D}$ and let (α,β) be a morphism such that $(\alpha,\beta) : f \to g$. Let (ι_A, ι_B) be the morphism mapping f to f such that $\iota_A : A \to A$ and $\iota_B : B \to B$ are the identity maps on objects A and B, respectively. Then $(\alpha,\beta)\circ(\iota_A,\iota_B) : f \to g$ and $(\alpha,\beta)\circ(\iota_A,\iota_B) = (\alpha,\beta)$. Similarly, for $(\gamma,\delta) : h \to f$ we have $(\iota_A,\iota_B)\circ(\gamma,\delta) : h \to f$ and $(\iota_A,\iota_B)\circ(\gamma,\delta) = (\gamma,\delta)$, so $1_{(\alpha,\beta)} = (\iota_A,\iota_B)$:



Definition I.7.2. Let $\mathcal{C}$ be a category and $\{A_i \mid i \in I\}$ a family of objects of $\mathcal{C}$. A *product* for the family $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$ together with a family of morphisms $\{\pi_i : P \to A_i \mid i \in I\}$ such that for any object B and family of morphisms $\{\varphi_i : B \to A_i \mid i \in I\}$, there is a unique morphism $\varphi : B \to P$ such that $\pi_i \circ \varphi = \varphi_i$ for all $i \in I$. The product P of $\{A_i \mid i \in I\}$ is denoted $P = \prod_{i \in I} A_i$ or $(P, \{\pi_i\})$.

Note. Notice that the product P is itself required to be an object in category $\mathcal{C}$. So a family of objects in a category may not have a product. Products *do exist* in several familiar categories. For example, a product of sets (the Cartesian product) is given in Theorem 0.5.2. In the next section, we'll deal with products of groups.

Note. For $i_1, i_2 \in I$ and family $\{A_i \mid i \in I\}$, we have that A_{i_1} and A_{i_2} are "parts" of $P = \prod_{i \in I} A_i$. We also need morphisms π_{i_1} and π_{i_2} such that $\pi_{i_1} : P \to A_{i_1}$ and $\pi_{i_2} : P \to A_{i_2}$. Finally we need for any $B \in \mathcal{C}$ and for any morphisms φ_{i_1} and φ_{i_2} such that $\varphi_{i_1} : B \to A_{i_1}$ and $\varphi_{i_2} : B \to A_{i_2}$, a unique morphism $\varphi : B \to P$ such that (1) $\pi_{i_1} \circ \varphi : B \to A_{i_1}$ and $\varphi_{i_1} : B \to A_{i_1}$ are the same, and (2) $\pi_{i_2} \circ \varphi : B \to A_{i_2}$ and $\varphi_{i_2} : B \to A_{i_2}$ are the same. In other words, the following diagram commutes:



In general, we require commutivity over a diagram containing all φ_i, π_i, and A_i (and this is required for *all* $B \in \mathcal{C}$).

Note. The following result shows that the construction of the product is independent of the morphisms π_i.

Theorem I.7.3. Let $\mathcal{C}$ be a category of objects and $\{A_i \mid i \in I\}$ a family of objects in $\mathcal{C}$. If $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of $\{A_i \mid i \in I\}$ then P and Q are equivalent.

Note. If in an abstract category we reverse the directions of all morphisms in a statement about the category (or, equivalently, if we reverse all the "arrows" in the diagrams which are required to commute) then we create a "dual" statement about the category. The following is the dual statement of the definition of a product.

Definition. Let $\mathcal{C}$ be a category of objects and $\{A_i \mid i \in I\}$ a family of objects in $\mathcal{C}$. A *coproduct* (or *sum*) of the family $\{A_i \mid i \in I\}$ is an object S in $\mathcal{C}$, together with a family of morphisms $\{\iota_i : A_i \to S \mid i \in I\}$ such that for any object B and family of morphisms $\{\psi_i : A_i \to B \mid i \in I\}$, there is a unique morphism $\psi : S \to B$ such that $\psi \circ \iota_i = \psi_i$ for all $i \in I$. We denote the coproduct as $\coprod_{i \in I} A_i$.

Note. In the definition of coproduct S, the mapping in product P, $\pi_i : P \to A_i$, is replaced with the mapping $\iota_i : A_i \to S$, $\varphi_i : B \to A_i$ is replaced with $\psi_i : A_i \to B$, and unique $\varphi : B \to P$ is replaced with unique $\psi : S \to B$. The following claim is the dual of Theorem I.7.3.

Theorem I.7.5. Let $\mathcal{C}$ be a category of objects and $\{A_i \mid i \in I\}$ a family of objects in $\mathcal{C}$. If $(S, \{\iota_i\})$ and $(S', \{\lambda_i\})$ are both coproducts for the family $\{A_i \mid i \in I\}$, then S and S' are equivalent.

Note. In many categories, the "objects" are sets or are sets with an added structure (such as groups). When this is the case, the morphisms can be considered as functions on sets. This leads us to "concrete categories."

Definition I.7.6. A *concrete category* is a category $\mathcal{C}$ together with a function σ that assigns to each object A of $\mathcal{C}$ a set $\sigma(A)$, called the *underlying set* of A, such that

(i) every morphism mapping $A \to B$ of category $\mathcal{C}$ is a function on the underlying sets $\sigma(A) \to \sigma(B)$;

(ii) the identity morphism of each object A of $\mathcal{C}$ is the identity function on the underlying set $\sigma(A)$;

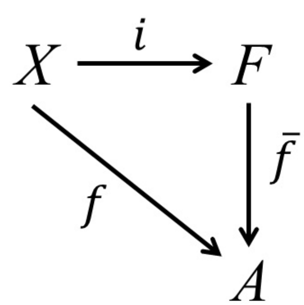
(iii) composition of morphisms in $\mathcal{C}$ agrees with composition of functions on the underlying sets.

Example. The category of groups where $\sigma(\langle G, * \rangle) = G$ is a concrete category.

Example. We saw in Example I.7.A that we can define a category using a group $\langle G, * \rangle$ with $\hom(G,G)$ is the elements of G and $a \circ b = a * b$ for $a, b \in G$. But this is not a concrete category since (i) of the definition of concrete category is violated. The morphisms are not *functions* on G but instead are *elements* of G.

Note. We will explore free groups in Section I.9. When we do, we will relate it to the following definition of "free" in the setting of categories.

Definition I.7.7. Let F be an object in a concrete category $\mathcal{C}$, X a nonempty set, and $i : X \to F$ a set map. Then object F is *free on the set* X provided that for any object A of $\mathcal{C}$ and set map $f : X \to A$, there exists a unique morphism of $\mathcal{C}$, $\overline{f} : F \to A$, such that $\overline{f} \circ i = f$ as a set map $X \to A$. This gives the mappings:



Example. Let $A = G$ be a group and $g \in G$. Let $F = \mathbb{Z}$. We want to find $\overline{f}$, a homomorphism from $\mathbb{Z}$ to G, $\overline{f} : \mathbb{Z} \to G$. If F is free on X, then $\overline{f}$ is uniquely determined by i and f. We choose X as a subset of $F = \mathbb{Z}$ (here, we choose $X = \{1\}$) and let i be the inclusion map (see page 4; it just maps $\{1\} \to \{1\} \subset \mathbb{Z}$). We need to define $f(1)$, say $f(1) = g$. Then (since 1 generates $\mathbb{Z}$) the homomorphism $\overline{f} : \mathbb{Z} \to G$ is uniquely determined and hence F *is* free on X. This example illustrates the value of having object F free on set X—to define a morphism on F, it suffices to define the morphism on $X = i(X) \subset F$ (where i is the inclusion map).

Theorem I.7.8. If $\mathcal{C}$ is a concrete category, if F and F' are objects of $\mathcal{C}$ such that F is free on the set X and F' is free on the set X' and $|X| = |X'|$, then F is equivalent to F'.

Note. To summarize this section up to this point, we have seen that any two products or coproducts for a given family of objects are actually equivalent (Theorems I.7.3 and I.7.5). Also, two objects free on the same set are equivalent (Theorem I.7.8). We now combine all these ideas in a single concept.

Definition I.7.9. An object I in a category $\mathcal{C}$ is *universal* (or *initial*) if for each object C of $\mathcal{C}$ there exists one and only one morphism mapping $I \to C$. An object T of $\mathcal{C}$ is *couniversal* (or *terminal*) if for each C of $\mathcal{C}$ there exists one and only one morphism mapping $C \to T$.

Note. As when we compare the definition of product and coproduct, the difference in the definitions of universal and couniversal involve reversals of the directions of mappings.

Theorem I.7.10. Any two universal (respectively, couniversal) objects in a category $\mathcal{C}$ are equivalent.

Example. In the category of all groups, $\langle e \rangle$ is universal since for any group G there is only one homomorphism (in this category, the morphisms are group homomorphisms) mapping $\langle e \rangle \to G$, namely the identity map on $\langle e \rangle$. Also, $\langle e \rangle$ is couniversal since the only homomorphism mapping $G \to \langle e \rangle$ is $f(g) = e$ for all $g \in G$.

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