Section I.8. Direct Products and Direct Sums

Note. In this section, we introduce general products of groups. This idea will be important when we classify finitely generated abelian groups in Section II.2.

Note. We defined the direct product of two groups in Section I.1. Semigroups, Monoids, and Groups We now extend this idea to a collection of groups indexed by an arbitrary index set I (possibly infinite or maybe even uncountable).

Definition. Consider an indexed family of groups $\{G_i \mid i \in I\}$. Define a binary operation on the Cartesian product of sets $\prod_{i\in I} G_i$ as follows. If $f, g \in \prod_{i\in I} G_i$ (that is, $f, g : I \to \cup_{i\in I} G_i$ and $f(i), g(i) \in G_i$ for all $i \in I$) then $fg : I \to \cup_{i\in I} G_i$ is the function given by $i \mapsto f(i)g(i) \in G_i$. So $fg \in \prod_{i\in I} G_i$ by Definition 0.5.1. The set $\prod_{i\in I} G_i$ together with this binary operation (where we identify $f \in \prod_{i\in I} G_i$ with its image, the ordered set $\{a_i \mid i \in I\}$) is the *direct product* (or *complete direct sum*) of the family of groups $\{G_i \mid i \in I\}$.

Note. In a crude sense, we can think of the elements of $\prod G_i$ as $|I|$-tuples in which the binary operation is performed componentwise. If $I = \{1, 2, \ldots, n\}$ is finite, then we denote $\prod G_i$ as $G_1 \times G_2 \times \cdots \times G_n$ (in multiplicative notation) or $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ (in additive notation).

Theorem I.8.1. If $\{G_i \mid i \in I\}$ is a family of groups, then

(i) the direct product $\prod G_i$ is a group,

(ii) for each $k \in I$, the map $\pi_k : \prod G_i \to G_k$ given by $f \mapsto f(k)$ is an epimorphism (i.e., an onto homomorphism) of groups.

The map π_k is called the *canonical projection* of the direct product.

Proof. A homework exercise.

Theorem I.8.2. Let $\{G_i \mid i \in I\}$ be a family of groups, let H be a group, and let $\{\varphi_i : H \to G_i \mid i \in I\}$ a family of group homomorphisms. Then there is a unique homomorphism $\varphi : H \to \prod G_i$ such that $\pi_i \varphi = \varphi_i$ for all $i \in I$ and this property determines $\prod G_i$ uniquely up to isomorphism. (In other words, $\prod G_i$ is a product in the category of groups.)

Note. If each G_i is abelian, then $\prod G_i$ is abelian (since the operation on $\prod G_i$ is calculated "component-wise" as given in the first definition of the notes for this section). So $\prod G_i$ is a product in the category of abelian groups.

Definition I.8.3. The *external weak direct product* of a family of groups $\{G_i \mid i \in I\}$, denoted $\prod_{i \in I}^{w} G_i$, is the set of all $f \in \prod_{i \in I} G_i$ such that $f(i) = e_i$ (where $e_i \in G_i$ is the identity element of G_i) for all but a finite number of $i \in I$. If all the groups G_i are additive abelian groups, then $\prod_{i \in I}^{w} G_i$ is called the *external direct sum* and is denoted $\sum_{i \in I} G_i$.

Note. Of course if I is finite, then there is no difference in the weak direct product and the direct product.

Theorem I.8.4. If $\{G_i \mid i \in I\}$ is a family of groups, then

(i) $\displaystyle\prod_{i\in I}^{w} G_i$ is a normal subgroup of $\prod_{i\in I} G_i$;

(ii) for each $k \in I$, the map $\iota_k : G_k \to \displaystyle\prod_{i\in I}^{w} G_i$ given by $\iota_k(a) = \{a_i\}_{i\in I}$ where $a_i = e_i$ for $i \neq k$ and $a_k = a$, is a monomorphism (one to one homomorphism) of groups;

(iii) for each $i \in I$, $\iota_i(G_i)$ is a normal subgroup of $\prod_{i\in I} G_i$.

Proof. Parts (ii) and (iii) are left as homework exercises.

Definition. The maps ι_k of Theorem I.8.4 are called the *canonical injections* of G_k into $\displaystyle\prod_{i\in I}^{w} G_i$ (or $\prod_{i\in I} G_i$).

Theorem I.8.5. Let $\{A_i \mid i \in I\}$ be a family of (additive) abelian groups. If B is an abelian group and $\{\psi_i : A_i \to B \mid i \in I\}$ is a family of homomorphisms, then there is a unique homomorphism mapping the external direct sum $\sum A_i$ to B, $\psi : \sum_{i\in I} A_i \to B$ such that $\psi\iota_i = \psi_i$ for all $i \in I$ and this property determines $\sum_{i\in I} A_i$ uniquely up to isomorphism. That is, $\sum_{i\in I} A_i$ is a coproduct in the category of abelian groups.

Note. Notice the use of the homomorphism properties of ψ and ξ in the proof of Theorem I.8.5 would not work unless the homomorphisms are applied to *finite* sums.

Note. Theorem I.8.5 is false if we remove the restriction that the groups are abelian. So the external weak direct product (or external direct sum) is not in general a coproduct in the category of all groups (as is to be shown in Exercise I.8.4).

Note I.8.A. The following result gives conditions under which a group is isomorphic to the weak direct product of a family of its subgroups (notice that we again see the use of normal subgroups). In Supplement. Direct Products and Semidirect Products, a result such as the following is called a "Recognition Theorem." See Theorem DF.5.9 in the supplement for a special case of the next result.

Theorem I.8.6. Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of a group G such that

(i) $G = \langle \cup_{i \in I} N_i \rangle$;

(ii) for each $k \in I$, we have $N_k \cap \langle \cup_{i \neq k} N_i \rangle = \langle e \rangle$.

Then $G \cong \prod_{i \in I}{}^{w} N_i$.

Corollary I.8.7. If $N_1, N_2, \ldots, N_r$ are normal subgroups of a group G such that $G = N_1 N_2 \cdots N_r$ and for each $1 \leq k \leq r$ we have $N_k \cap (N_1 \cdots N_{k-1} N_{k+1} \cdots N_r) = \langle e \rangle$ then $G \cong N_1 \times N_2 \times \cdots \times N_r$.

Proof. This follows from Theorem I.8.6 when we observe that $\langle N_1 \cup N_2 \cup \cdots \cup$

$N_r\rangle = N_1 N_2 \cdots N_r = \{n_1 n_2 \cdots n_r \mid n_i \in N_i\}$ by Theorem I.5.3 (plus mathematical induction). ∎

Definition I.8.8. Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of a group G such that $G = \langle \cup_{i\in I} N_i\rangle$ and for each $k \in I$ we have $N_k \cap \langle \cup_{i\neq k} N_i\rangle = \langle e\rangle$. Then G is an *internal weak direct product* of the family $\{N_i \mid i \in I\}$ (or the *internal direct sum* if G is additive and abelian).

Note I.8.B. Notice the difference between the terminology for multiplicative versus additive groups. The terms "direct product" and "complete direct sum" correspond; the terms "internal weak direct product" and "internal direct sum" correspond. Notice the funny use of "complete" in the sum setting and the use of "weak" in the multiplicative setting so that there are no "complete products" nor "weak sums."

Note I.8.C. Notice that when we speak of an internal weak direct product that, by definition, the groups $\{N_i \mid i \in I\}$ are normal subgroups of G. When dealing with an internal weak direct product of a finite number of groups (which we will do in Section II.3 when addressing indecomposable groups), we will use a particular notation (not in Hungerford). If G is the internal direct product of normal subgroups $N_1, N_2, \ldots, N_n$ then we write $G = N_1 \times^i N_2 \times^i \cdots \times^i N_n$. With this notation, we have elements of $H \times K$ as ordered pairs, but the elements of $H \times^i K$, where H and K are normal subgroups of G, are elements of G. Notice that for a finite collection of groups, we need not distinguish between "weak direct product" and "direct product."

Note. The following (a corollary of Theorem I.8.6) classifies internal direct products.

Theorem I.8.9. Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of a group G. G is the internal weak direct product of the family $\{N_i \mid i \in I\}$ if and only if every nonidentity element of G is a unique product $a_{i_1}a_{i_2}\cdots a_{i_n}$ with $i_1, i_2, \ldots, i_n$ distinct elements of I and $e \neq a_{i_k} \in N_{i_k}$ for each $k = 1, 2, \ldots, n$.

Proof. A homework exercise.

Note I.8.D. Exercise I.8.11 also gives a necessary and sufficient condition for G to be the internal weak direct product of a collection of subgroups: "Let $\{N_i \mid i \in I\}$ be a family of subgroups of a group G. Then G is the internal weak direct product of $\{N_i \mid i \in I\}$ if and only if:

(i) $a_ia_j = a_ja_i$ for all $i \neq j$ and $a_i \in N_i$, $a_j \in N_j$;

(ii) every nonidentity element of G is uniquely a product $a_{i_1}a_{i_2}\cdots a_{i_n}$, where $i_1, i_2, \ldots, i_n$ are distinct elements of I and $e \neq a_{i_k} \in H_{i_k}$ for each k."

Note I.8.E. There is a subtle difference between internal and external weak direct products. For G an internal weak direct product of groups N_i we have by definition that each N_i is a subgroup of G and G is *isomorphic* to the external direct product $\prod^w N_i$ by Theorem I.8.6. But the external direct product does not contain groups N_i but contains *isomorphic* copies of them. Elements of $\prod^w N_i$ are "$|I|$-tuples." The $|I|$-tuples with each entry e, except that the ith entries range over the elements of N_i, forms a subgroup of $\prod^w N_i$ which is isomorphic to N_i. This subgroup is $\iota_i(N_i)$.

Example. To be more specific, it is easily confirmed that $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Denote the elements of these groups as: $\mathbb{Z}_2 = \{\overline{0}_2, \overline{1}_2\}$, $\mathbb{Z}_3 = \{\overline{0}_3, \overline{1}_3, \overline{2}_3\}$, and $\mathbb{Z}_6 = \{\overline{0}_6, \overline{1}_6, \overline{2}_6, \overline{3}_6, \overline{4}_6, \overline{5}_6\}$ (where the subscript indicates the congruence relation on $\mathbb{Z}$ determining the equivalence classes). We have external direct sum $\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \{(\overline{0}_2, \overline{0}_3), (\overline{0}_2, \overline{1}_3), (\overline{0}_2, \overline{2}_3), (\overline{1}_2, \overline{0}_3), (\overline{1}_2, \overline{1}_3), (\overline{1}_2, \overline{2}_3)\}$ and $f : \mathbb{Z}_6 \to \mathbb{Z}_2 \oplus \mathbb{Z}_3$ defined as:

$$f(\overline{0}_6) = (\overline{0}_2, \overline{0}_3),\ \ f(\overline{1}_6) = (\overline{1}_2, \overline{2}_3),\ \ f(\overline{2}_6) = (\overline{0}_2, \overline{1}_3),$$

$$f(\overline{3}_6) = (\overline{1}_2, \overline{0}_3),\ \ f(\overline{4}_6) = (\overline{0}_2, \overline{2}_3),\ \ f(\overline{5}_6) = (\overline{1}_2, \overline{1}_3)$$

is an isomorphism. To write $\mathbb{Z}_6$ as an *internal* direct product, we take the subgroups $H = \{\overline{0}_6, \overline{3}_6\}$ and $K = \{\overline{0}_6, \overline{2}_6, \overline{4}_6\}$ of $\mathbb{Z}_6$. Then

$$\langle H \cup K \rangle = \langle \{\overline{0}_6, \overline{2}_6, \overline{3}_6, \overline{4}_6\} \rangle = \mathbb{Z}_6$$

because $\overline{1}_6 = \overline{3}_6 + \overline{4}_6$ and $\overline{5}_6 = \overline{2}_6 + \overline{3}_6$. So in our notation we have $\mathbb{Z}_6 = H \times^i K$.

Notice that Hungerford uses the notation "$\prod_{i \in I}^w N_i$" for an *external* weak direct product, but he doesn't give a unique notation for an *internal* weak direct product. He comments (see page 62): "Practically speaking, this distinction is not very important and the adjectives 'internal' and 'external' will be omitted whenever no confusion is possible." He then says that we simply use the notation for an external direct product to represent both an external and internal direct product: $G = \prod_{i\in I}^{w} N_i$. In these class notes, we will try to be a bit more careful than Hungerford.

Note I.8.F. In Exercise I.8.B, you are asked to prove that if $G = H \times K$, then $\{e_H\} \times K$ is a normal subgroup of G and $G/(\{e_H\} \times K) = (H \times K)/(\{e_H\} \times K) \cong H$ (and similarly, $H \times \{e_K\}$ is a normal subgroup of G and $G/(H \times \{e_K\}) = (H \times K)/(K \times \{e_K\} \cong K))$. So if G is the *external* direct product of H and K (notationally, $G = H \times K$), then H and K are isomorphic to normal subgroups of G by this exercise. If G is the *internal* direct product of H and K (notationally $G = H \times^i K$), then H and K are normal subgroups *by the definition* of "internal direct product." More specifically, in Exercise I.8.C you are asked to prove that if G is the internal direct product of normal subgroups N_1 and N_2, $G = N_1 \times^i N_2$, then $G \cong N_1 \times N_2$, and conversely if $G \cong H \times K$, then there are subgroups N_1 and N_2 of G such that $N_1 \cong H$, $N_2 \cong K$, and $G = N_1 \times^i N_2$. So all this (even in the more general case of arbitrary weak products, instead of just a product of two groups) can be whitewashed with a blanket "isomorphism" comment; thus Hungerford's approach.

Theorem I.8.10. Let $\{f_i : G_i \to H_i \mid i \in I\}$ be a family of homomorphisms of groups and let $f = \prod f_i$ be the mapping of $\prod G_i \to \prod H_i$ given by $\{a_i\} \mapsto \{f_i(a_i)\}$. Then f is a homomorphism of groups such that $f(\prod^w G_i) \subset \prod^w H_i$, $\mathrm{Ker}(f) = \prod \mathrm{Ker}(f_i)$, and $\mathrm{Im}(f) = \prod \mathrm{Im}(f_i)$ (where all products are over $i \in I$). Consequently f is a monomorphism (or epimorphism) if and only if each f_i is a monomorphism (or epimorphism).

Proof. A homework exercise.

Corollary I.8.11. Let $\{G_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ be families of groups such that N_i is a normal subgroup of G_i for each $i \in I$.

(i) $\prod N_i$ is a normal subgroup of $\prod G_i$ and $\prod G_i / \prod N_i \cong \prod G_i/N_i$.

(ii) $\prod^w N_i$ is a normal subgroup of $\prod^w G_i$ and $\prod^w G_i / \prod^w N_i \cong \prod^w G_i/N_i$.

Note I.8.G. Corollary I.8.11 gives a nice behavior of products and quotients of groups. It might be tempting to think of the process of taking a group product and a quotient group as somehow inverses of each other (especially in light of Exercise I.8.B; see Note I.8.F above). However, this is not the case; at least not in the following sense. It is **not** in general **true** for $N \triangleleft G$ that we have $(G/N) \times N \cong G$. Consider, for example, the easy example of $G = \mathbb{Z}_4$ and $N = \{\overline{0}, \overline{2}\} \cong \mathbb{Z}_2$. Then $N \triangleleft G$ but $G/N = \mathbb{Z}_4/\{\overline{0}, \overline{2}\} \cong \mathbb{Z}_2$ (since the quotient group is of order 2) and so $(G/N) \times N \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = V$ (the Klein 4-group). Of course $G = \mathbb{Z}_4 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 = (G/N) \times N$.