Section II.1. Free Abelian Groups

Note. This section and the next, are independent of the rest of this chapter. The primary use of the results of this chapter is in the proof of the Fundamental Theorem of Finitely Generated Abelian Groups (Theorem II.2.1). This is also covered in Introduction to Modern Algebra 2 (MATH 4137/5137); see my online notes on Section 38. Free Abelian Groups. However, the introductory class only presents free abelian groups with a finite basis. Hungerford gives more general results.

Note. Throughout this section, since we are discussing abelian groups, we use additive notation. So when we write "na" for $n \in \mathbb{N}$ and $a \in G$, we mean $a + a + \cdots + a$ (n-times). For $-n \in \mathbb{N}$, "na" denotes $(-a) + (-a) + \cdots + (-a)$ (n-times).

Definition. For group G and set $X \subset G$, a *linear combination* of elements of X is a sum $n_1x_1 + n_2x_2 + \cdots + n_kx_k$ where $n_i \in \mathbb{Z}$ and $x_i \in X$. A *basis* of an abelian group F is a subset X of F such that

(i) $F = \langle X \rangle$, and

(ii) for distinct $x_1, x_2, \ldots, x_n \in X$ and $n_i \in \mathbb{Z}$, $n_1x_1 + n_2x_2 + \cdots + n_kx_k = 0$ implies $n_i = 0$ for all i.

Note 2.1.A. We will see that every basis of a "free abelian group" is of the same cardinality (Theorem II.1.2). So a basis of a group seems to be almost identical to a basis of a vector space. However, in Exercise II.1.2(b) it is shown that a linearly independent set of elements which is the same size as a (finite) basis may not be a basis. So the basis of an abelian group is a different concept.

Theorem II.1.1. The following conditions on an abelian group F are equivalent.

(i) F has a nonempty basis.

(ii) F is the (internal) direct sum of a family of infinite cyclic subgroups.

(iii) F is (isomorphic to) a direct sum of copies of the additive group $\mathbb{Z}$ of integers.

(iv) There exists a nonempty set X and a function $\iota : X \to F$ with the following property: Given an abelian group G and function $f : X \to G$, there exists a unique homomorphism of groups $\overline{f} : F \to G$ such that $\overline{f}\iota = f$. In other words, F is a free object in the category of abelian groups.

Definition. An abelian group F that satisfies the conditions of Theorem II.1.1 is a *free abelian group* (on the set X).

Example. In Exercise II.1.11(b) it is shown that the positive rationals $\mathbb{Q}^*$ under multiplication form a free abelian group with basis $X = \{p \in \mathbb{N} \mid p \text{ is prime}\}$.

Note 2.1.B. Theorem II.1.1 shows how to construct a free abelian group F with basis X for any given set X. Just let F be the direct sum $\sum \mathbb{Z}$ where the $\mathbb{Z}$'s are indexed by set X. As shown in the proof of (iii) implies (i), $\{\theta_x \mid x \in X\}$ (as defined in the proof) is a basis of $F = \sum \mathbb{Z}$ and F is free on the set $\{\theta_x \mid x \in X\}$. The cyclic subgroup $\langle \theta_x \rangle = \{n\theta_x \mid n \in \mathbb{Z}\} = \mathbb{Z}\theta_x$ is denoted $\langle x \rangle = \mathbb{Z}x$. In this notation $F = \sum_{x \in X} \langle \theta_x \rangle$ is written $F = \sum_{x \in X} \mathbb{Z}x$ and an element of F has the form $n_1x_1 + n_2x_2 + \cdots + n_kx_k$ where $n_i \in \mathbb{Z}$ and $x_i \in X$. That is, X is a basis of F (in this notation).

Example. The group $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (r times) is a free abelian group with basis $\{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)\}$. The group $\mathbb{Z}_n$ is not a free abelian group since $nx = 0$ for every $x \in \mathbb{Z}_n$, contradicting property (ii) of the definition of basis of an abelian group.

Note. For the proof of the following, we need some cardinality results from Section 0.8.

Theorem II.1.2. Any two bases of a free abelian group F have the same cardinality.

Definition. The cardinal number of any basis X of the free abelian group F is the *rank* of F.

Note. I would call the following result **The Fundamental Theorem of Free Abelian Groups**! Similar to the Fundamental Theorem of Finite/Infinite Dimensional Vector Spaces, it classifies free abelian groups according to their rank (as opposed to the dimension of a vector space). For details on the vector space results, see my class notes for Fundamentals of Functional Analysis (MATH 5740) online on:

1. For the Fundamental Theorem of Finite Dimensional Vector Spaces, see Supplement. Groups, Fields, and Vector Spaces,

2. For the Fundamental Theorem of Infinite Dimensional Vector Spaces, see Supplement. Projections and Hilbert Space Isomorphisms.

Proposition II.1.3. Let F_1 be the free abelian group on the set X_1 and F_2 the free abelian group on the set X_2. Then $F_1 \cong F_2$ if and only if F_1 and F_2 have the same rank (that is, $|X_1| = |X_2|$).

Note. We can drop "abelian" from the statement of Proposition II.1.3 to prove that: If F is a free group on set X and G is a free group on set Y, then $F \cong G$ if and only if $|X| = |Y|$. See Exercise II.1.12.

Note. The following is analogous to Corollary I.9.3 (which states that every group is the homomorphic image of a free group).

Theorem II.1.4. Every abelian group G is the homomorphic image of a free abelian group of rank $|X|$, where X is a set of generators of G.

Note. Before we prove the main result which we will use in the proof of the Fundamental Theorem of Finitely Generated Abelian Groups, we need a preliminary result.

Lemma II.1.5. If $\{x_1, x_2, \ldots, x_n\}$ is a basis of a free abelian group F and $a \in \mathbb{Z}$, then for all $i \neq j$, $\{x_1, x_2, \ldots, x_{j-1}, x_j + ax_i, x_{j+1}, x_{j+2}, \ldots, x_n\}$ is also a basis of F.

Theorem II.1.6. If F is a free abelian group of finite rank n and G is a nonzero subgroup of F, then there exists a basis $\{x_1, x_2, \ldots, x_n\}$ of F, an integer r (where $1 \leq r \leq n$) and positive integers $d_1, d_2, \ldots, d_r$ such that $d_1 \mid d_2 \mid \cdots \mid d_r$ (that is, $d_1 \mid d_2$, $d_2 \mid d_3$, $\ldots$, $d_{r-1} \mid d_r$) and G is free abelian with basis $\{d_1x_1, d_2x_2, \ldots, d_rx_r\}$.

Note. To illustrate Theorem II.1.6, consider a free abelian group of rank 2, $F = \mathbb{Z} \oplus \mathbb{Z}$. Then $G < F$ where $G = 2\mathbb{Z} \oplus 3\mathbb{Z}$. Now, of course, $\{(1,0),(0,1)\}$ is a basis of F and $\{2(1,0), 3(0,1)\} = \{(2,0),(0,3)\}$ is a basis for G, but neither $2 \mid 3$ nor $3 \mid 2$. So we need a different basis for F. Consider $\{x_1, x_2\} = \{(2,3),(1,2)\}$. Then $\{x_1, x_2\}$ is linearly independent, $2(2,3) - 3(1,2) = (1,0)$ and $-(2,3) + 2(1,2) = (0,1)$. So $\langle\{x_1, x_2\}\rangle = F$. Now we claim $1(2,3) = d_1x_1$ and $6(1,2) = (6,12) = d_2x_2$ form a basis of $G = 2\mathbb{Z} \oplus 3\mathbb{Z}$. Notice that $1 \mid 6$ so $d_1 \mid d_2$. Now $\{(2,3),(6,12)\}$ is a linearly independent set. Also $4(2,3) - (6,12) = (2,0)$ and $-3(2,3) + (6,12) = (0,3)$, so $\{(2,3),(6,12)\} = \{d_1x_1, d_2x_2\}$ is a basis of $G = 2\mathbb{Z} \oplus 3\mathbb{Z} < F$, as claimed in Theorem II.1.6.

Corollary II.1.7. If G is a finitely generated abelian group generated by n elements, then every subgroup H of G may be generated by m elements with $m \leq n$.

Note. Corollary II.1.7 does not hold if we remove the constraint "abelian," as is shown by example in Exercise II.1.8.

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