Section II.4. The Action of a Group on a Set

Note. In this section, we introduce the idea of a group "acting" on a set. This has applications to counting (see Fraleigh's Section III.17 "Application of G-Sets to Counting"), but our main use of this topic is to the proofs of the Sylow Theorems in the next section.

Definition II.4.1. An *action* of a group G on a set S is a function mapping $G \times S \to S$ (denoted $(g, x) \mapsto g \star x$) such that for all $x \in S$ and $g_1, g_2 \in G$ we have

$$e \star x = x \text{ and } (g_1 g_2) \star x = g_1 \star (g_2 \star x).$$

When this occurs, we say that G *acts on set* S.

Note. Hungerford denotes the action of group element g on set element x simply as "gx." This is common and is followed by Fraleigh, and Dummit and Foote. However, in these class notes we use a star, $\star$, to denote this: $g \star x$. This is not common and simply introduced here to distinguish group action from the binary operation in a group (since many applications of group action will involve set S as a group itself).

Example. Consider the symmetry group S_n and the set $I_n = \{1, 2, \ldots, n\}$. For $\sigma \in S_n$ and $x \in I_n$, consider the function mapping $S_n \times I_n \to I_n$ defined as $\sigma \star x = \sigma(x)$. Since $\iota \star x = \iota(x) = x$ and $(\sigma_1 \sigma_2) \star x = (\sigma_1 \circ \sigma_2)(x) = \sigma_1(\sigma_2(x)) = \sigma_1 \star (\sigma_2 \star x)$ then S_n acts on a set I_n. Similarly, the alternating group A_n acts on set I_n. The dihedral group D_n also acts on set I_n when we interpret the elements of D_n as acting on the set of vertices of an n-gon with labels $1, 2, \ldots, n$ (labeled clockwise, say).

Example. Let G be a group and H a subgroup of G. Then we can treat group H as acting on set G by defining function mapping $H \times G \to G$ as $h \star x = hx$ where hx is the product in group G. Certainly the identity of H behaves as required and associativity of the binary operation gives the second condition in the definition of action. The action of $h \in H$ on G is called *left translation* (it is also a *permutation* of G). If K is another subgroup of G and S is the set of all left cosets of K in G, then H acts on the set of cosets S by left translation, as given by $h \star xK = (hx)K$.

Example. Let H be a subgroup of a group G. An action of H on set G is given by the mapping $H \times G \to G$ as $h \star x = hxh^{-1}$. Of course $e \star x = exe^{-1} = x$ and

$$(g_1g_2)\star x = g_1g_2x(g_1g_2)^{-1} = g_1g_2xg_2^{-1}g_1^{-1} = g_1(g_2xg_2^{-1})g_1^{-1} = g_1(g_2\star x)g_1^{-1} = g_1\star(g_2\star x).$$

This action of $h \in H$ on G is called *conjugation* by h and element $hxh^{-1} \in G$ is a *conjugate* of x. If K is any subgroup of G and $h \in H$, then hKh^{-1} is a subgroup of G isomorphic to K by Exercise I.5.6. So H acts on the set S of all subgroups of G by conjugation: $h \star K = hKh^{-1}$ (which is an action by the argument above). The group hKh^{-1} is said to be *conjugate* to K.

Theorem II.4.2. Let G be a group that acts on a set S.

(i) The relation on S defined by

$$x \sim x' \iff g \star x = x' \text{ for some } g \in G$$

is an equivalence relation on set S.

(ii) For each $x \in S$, $G_x = \{g \in G \mid g \star x = x\}$ is a subgroup of G.

Note/Definition. We know that the equivalence classes of an equivalence relation partition the set on which they are based (Theorem 0.4.1). The equivalence classes of the equivalence relation given in Theorem II.4.2(i) are the *orbits* of G on S. The orbit of a given $x \in S$ is denoted $\overline{x}$. The subgroup G_x is called the *stabilizer* of x (or the *subgroup fixing* x, or the *isotropy group* of x).

Definition. If a group G acts on itself by conjugation, then the orbit $\{gxg^{-1} \mid g \in G\}$ of $x \in G$ is the *conjugacy class* of x. If a subgroup H acts on G by conjugation then the stabilizer group $H_x = \{h \in H \mid hxh^{-1} = x\} = \{h \in H \mid hx = xh\}$ is the *centralizer of x in H*, denoted $C_H(x)$. If $H = G$, $C_G(x)$ is simply called the *centralizer of x*. If H acts by conjugation on the set S of all subgroups of G, then the subgroup of H fixing $K \in S$, namely $\{h \in H \mid hKh^{-1} = K\}$ is the *normalizer of K in H*, denoted $N_H(K)$. The group $N_G(K)$ is the *normalizer of K*.

Theorem II.4.3. If a group G acts on a set S, then the cardinal number of $x \in S$, $|\overline{x}|$, is the index $[G : G_x]$ (recall that $[G : G_x]$ is the cardinal number of the left cosets of subgroups G_x in group G).

Note. We now consider the special case of action on a set where the action is conjugation and the set is a group.

Corollary II.4.4. Let G be a finite group and K a subgroup of G.

(i) The number of elements in the conjugacy class of $x \in G$ is $[G : C_G(x)]$, which divides $|G|$.

(ii) If $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$ are the distinct conjugacy classes of G, then $|G| = \sum_{i=1}^{n} [G : C_G(x_i)]$.

(iii) The number of subgroups of G conjugate to K is $[G : N_G(K)]$, which divides $|G|$.

Definition. For finite group G, the equation $|G| = \sum_{i=1}^{n} [G : C_G(x_i)]$ given in Corollary II.4.4(ii) is the *class equation* of group G.

Theorem II.4.5. If a group G acts on set S, then this action induces a homomorphism mapping $G \to A(S)$ where $A(S)$ is the group of all permutations of S.

Corollary II.4.6. Cayley's Theorem.

If G is a group, then there is a monomorphism (a one to one homomorphism) mapping $G \to A(G)$. Hence, every group is isomorphic to a group of permutations. In particular, every finite group is isomorphic to a subgroup of S_n with $n = |G|$.

Note. Recall that if G is a group, then the set of all automorphisms of G (that is, isomorphisms of G with itself) is a group, denoted $\mathrm{Aut}(G)$ (see Exercise I.2.15(a)).

Corollary II.4.7. Let G be a group.

(i) For each $g \in G$, conjugation by g induces an automorphism of G.

(ii) There is a homomorphism mapping $G \to \mathrm{Aut}(G)$ whose kernel is $C(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$.

Definition. The automorphism τ_g of G in Corollary II.4.7(i) (which maps $x \in G$ to gxg^{-1}) is the *inner automorphism induced by g*. The set $C(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$ is the *center* of G.

Note II.4.A. The center of G, $C(G)$, is a normal subgroup of G by Theorem I.5.1(iii). If $g \in C(G)$ then $g = xgx^{-1}$ for all $x \in G$ and so the conjugacy class of g consists of only g. So if G is finite and $x \in C(G)$ then by Corollary II.4.4(i) $[G : C_G(x)] = 1$. So the class equation of Corollary II.4.4(ii) is

$$\begin{aligned} |G| &= \sum_{i=1}^{n} [G : C_G(x_i)] \text{ where the distinct conjugacy classes of } G \text{ are } \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n \\ &= |C(G)| + \sum_{i=1}^{m} [G : C_G(x_i)] \end{aligned}$$

where $\overline{x}_1, \overline{x}_2, \ldots \overline{x}_m$ are the distinct conjugacy classes of G for which $[G : C_G(x_i)] > 1$ (and so $x_i \in G \setminus C(G)$).

Proposition II.4.8. Let H be a subgroup of a group G and let G act on set S of all left cosets of H in G by left translation. Then the kernel of the induced homomorphism mapping $G \to A(S)$ is contained in H.

Corollary II.4.9. If H is a subgroup of index n in a group G (that is, H has n left cosets in G) and no nontrivial normal subgroup of G is contained in H, then G is isomorphic to a subgroup of S_n.

Note. The following result foreshadows the Sylow Theorems of the next section.

Corollary II.4.10. If H is a subgroup of a finite group G of index p (that is, H has p left cosets in G), where p is the smallest prime dividing the order of G, then H is normal in G.

Revised: 4/12/2021