Section II.6. Classification of Finite Groups

Note. In this section, based largely on Sylow's three theorems, we classify all groups of order up to 15. First, we classify groups of order pq where p and q are distinct primes.

Proposition II.6.1. Let p and q be primes such that $p > q$.

(i) If $q \nmid p-1$ then every group of order pq is isomorphic to the cyclic group $\mathbb{Z}_{pq}$.

(ii) If $q \mid p-1$ then there are (up to isomorphism) exactly two distinct groups of order pq: the cyclic group $\mathbb{Z}_{pq}$ and a nonabelian group K generated by elements c and d such that these elements have orders $|c| = p$ and $|d| = q$. Also $dc = c^s d$ where $s \not\equiv 1 \pmod{p}$ and $s^q \equiv 1 \pmod{p}$. This nonabelian group is called a *metacyclic group*.

Note. With $C_p = \langle c \rangle$ and $C_q = \langle d \rangle$ as cyclic multiplicative groups of orders p and q, respectively, and $\theta : C_q \to \mathrm{Aut}(C_p)$ given by $\theta(d^i) = \alpha^i$ where $\alpha : C_p \to C_p$ is given by the automorphism mapping $c^i \mapsto c^{si}$, then the metacyclic group is isomorphic to the semidirect product $C_p \times_\theta C_q$ (see Exercises II.6.1 and II.6.2).

Corollary II.6.2. If p is an odd prime, then every group of order $2p$ is isomorphic either to the cyclic group $\mathbb{Z}_{2p}$ or the dihedral group D_p.

Note. We can now classify some finite groups. If G is a group of prime order p, then by Lemma II.2.3 $G \cong \mathbb{Z}_p$. If G is a group of order $2p$ where p is an odd prime, then Corollary II.6.2 gives the two possibilities for G.

Proposition II.6.3. There are (up to isomorphism) exactly two distinct nonabelian groups of order 8: the quaternion group Q_8 (see Exercise I.2.3) and the dihedral group D_4.

Proposition II.6.4. There are (up to isomorphism) exactly three distinct nonabelian groups of order 12: the dihedral group D_6, the alternating group A_4, and a group T generated by elements a and b such that $|a| = 6$, $b^2 = a^3$, and $ba = a^{-1}b$.

Note. The group T of order 12 is an example of a *dicyclic group*. A presentation of the nth dicyclic group, denoted Dic_n, is given by $(X \mid Y)$ where $X = \{a, b\}$ and $Y = \{a^{2n}, a^n b^{-2}, b^{-1}aba\}$. That is, Dic_n is generated by a and b where a and b satisfy the relations $a^{2n} = e$, $a^n = b^2$, and $b^{-1}ab = a^{-1}$. The group Dic_n is of order $4n$ (see Steven Roman's *Fundamentals of Group Theory: An Advanced Approach*, Springer-Verlag (2011), pages 347–348). So the group T is actually the third dicyclic group, Dic_3. The first dicyclic group is isomorphic to $\mathbb{Z}_4$; for $n \geq 2$, Dic_n is nonabelian. In fact, the second dicyclic group is isomorphic to the quaternions, $Q_8 \cong \text{Dic}_2$ (see Example I.9.A in the class notes and notice that $b^{-1}ab = a^{-1}$ implies that $ba = a^{-1}b$). When n is a power of 2, Dic_n is isomorphic to a "generalized quaternion group."

Note. "There is no known formula giving the number of distinct [i.e., nonisomorphic] groups of order n, for every n" (Hungerford, page 98). However we have the equipment to classify all groups of orders less than or equal to 15. For prime orders 2, 3, 5, 7, 11, and 13, Exercise I.4.3 tells us that there is only one group of each of these orders. By Corollary II.6.2, for orders 6, 10, and 14 there are two nonisomorphic groups of these orders n, namely $\mathbb{Z}_n$ and $D_{n/2}$. Exercise I.4.5 gives two groups of order 4, $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Theorem II.2.1 and Proposition II.6.3 classify the five groups of order 8, $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, Q_8, and D_4. Exercise II.5.13 and Theorem II.2.1 classify the two groups of order 9, $\mathbb{Z}_9$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. Theorem II.2.1 and Proposition II.6.4 classify the five groups of order 12 as $\mathbb{Z}_{12}$, $\mathbb{Z}_6 \oplus \mathbb{Z}_2$, A_4, D_6, and T. Proposition II.6.1 classifies the one group of order 15 as $\mathbb{Z}_{15}$.

Note. To summarize, the first table below gives the references to justify the second table which includes a list of all groups of order 15 or less (yes, yes, "up to isomorphism").

Order	**Reference**	**Order**	**Reference**
1	trivial	**9**	Exercise II.5.13
2	Exercise I.4.3		Theorem II.2.1
3	Exercise I.4.3	**10**	Corollary II.6.2
4	Exercise I.4.5	**11**	Exercise I.4.3
5	Exercise I.4.3	**12**	Theorem II.2.1
6	Corollary II.6.2		Proposition II.6.4
7	Exercise I.4.3	**13**	Exercise I.4.3
8	Theorem II.2.1	**14**	Corollary I.6.2
	Proposition II.6.3	**15**	Proposition II.6.1

Order	**Group**	**Comments**
1	$\mathbb{Z}_1$	The trivial group.
2	$\mathbb{Z}_2$	
3	$\mathbb{Z}_3 \cong A_3$	
4	$\mathbb{Z}_4$	
	Klein 4-group $V \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$	The smallest non-cyclic group.
5	$\mathbb{Z}_5$	
6	$\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$	
	$S_3 \cong D_3$	The smallest nonabelian group.
7	$\mathbb{Z}_7$	
8	$\mathbb{Z}_8$	
	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	
	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	
	D_4	Nonabelian.
	Quaternions Q_8	Nonabelian.
9	$\mathbb{Z}_9$	
	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	
10	$\mathbb{Z}_{10} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_5$	
	D_5	Nonabelian.
11	$\mathbb{Z}_{11}$	
12	$\mathbb{Z}_{12} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_4$	
	$\mathbb{Z}_2 \oplus \mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	
	$D_6 \cong \mathbb{Z}_2 \times D_3$	Nonabelian.
	A_4	Nonabelian; Smallest group which shows converse of Lagrange's Theorem does not hold.
	$\text{Dic}_3 \cong T$	Nonabelian, dicyclic group of order 12.
13	$\mathbb{Z}_{13}$	
14	$\mathbb{Z}_{14} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_7$	
	D_7	Nonabelian.
15	$\mathbb{Z}_{15} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_5$	

Note. There are 14 groups of order 16, 5 of which are abelian. A proof of this is given in Marcel Wild's "The Groups of Order Sixteen Made Easy," *American Mathematical Monthly*, **112**, January 2005, 20-31. The paper is "easy" since it does not use Sylow theorems, p-groups, generators/relations, nor group action. Many of the groups (namely, half of them) involve products (or semi-direct products; see Exercise II.6.1) of $\mathbb{Z}_2$ and a group of order 8.

Note. If we wish to go beyond order 16, then we know that there is only one group of order 17 since 17 is prime (and similarly for 19). By Proposition II.6.1, there are two groups of order 21, $\mathbb{Z}_{21}$ and another (nonabelian) group of order 21. Notice that this other group is a bit of a "mystery group"; it is not a symmetry group S_n, alternating group A_n, dihedral group D_n, nor a dicyclic group Dic_n (based on the orders of these groups). For order 22, we can apply Corollary II.6.2 to see that there are two groups, $\mathbb{Z}_{22}$ and D_{11} (and similarly for order 26). A webpage listing all groups of order at most 100 is online at "Tribimaximal Mixing From Small Groups" by K. Parattu, A. Wingerter (accessed 11/14/2019). See also The GAP Small Groups Library (accessed 11/14/2019) which contains results from the computational group theory software "GAP" (Groups, Algorithms, Programming).

Note. Three general results are worth mentioning. Let p be prime.

(1) We know that there is a single group (up to isomorphism) of order p and it is $\mathbb{Z}_p$ by Exercise I.4.3.

(2) There are only two groups of order p^2 for p prime, $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \oplus \mathbb{Z}_p$ (by Exercise II.5.13 a group of order p^2 is abelian, and so the Fundamental Theorem of Finitely Generated Abelian Groups [Theorem II.2.6] applies to give this result).

(3) For p^3, there are three abelian groups of order p^3 (by the Fundamental Theorem of Finitely Generated Abelian Groups), namely $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$, $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$, and $\mathbb{Z}_{p^3}$. When $p = 2$, we know that there are two nonabelian groups of order 8 by Proposition II.6.3, namely the quaternions Q_8 and the dihedral group D_4. For p an odd prime, there are two nonabelian groups of order p^3 as shown in Exercise II.6.8. One of these groups has generators a and b where $|a| = p^2$, $|b| = p$, and $b^{-1}ab = a^{1+p}$. The other has generators a, b, and c where $|a| = |b| = |c| = p$, $c = a^{-1}b^{-1}ab$, $ca = ac$, and $cb = bc$. Keith Conrad of the University of Connecticut describes these groups online: "Groups of Order p^3" (accessed 11/14/2019). The second type of group is an example of a *Heisenberg group* and is of the form:

$$\text{Heis}(\mathbb{Z}_p) = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, a, b, c \in \mathbb{Z}_p \right\}.$$

The entries a, b, c can be taken from other algebraic structures. If they are real numbers, then what is produced is the *continuous Heisenberg group* which has applications in one dimensional quantum mechanical systems (hence the connection with physicist Werner Heisenberg, 1901–1976).

Note. See also Supplement. Small Groups, notes which I use in Introduction to Modern Algebra (MATH 4127/5127). This document overlaps with some of the material in this section of notes, but also includes a brief discussion of the "Monster group" and a reference to the $\mathbb{ATLAS}$ *of Finite Groups*, by Conway, Curtis, Norton, Parker, and Wilson (Oxford University Press, 1985).

Revised: 5/1/2021