Section II.7. Nilpotent and Solvable Groups

Note. In this section, we define nilpotent and solvable groups and see that every nilpotent group is solvable (Proposition II.7.10). Solvable groups realize their greatest importance in the proof of the insolvability of the quintic in Chapter V (in the Appendix to Section V.9).

Note. Hungerford describes our success with finite abelian groups (Section II.2) and p-groups (Section II.5) as yielding "striking results." In order to combine these results, we consider the class of finite groups which are products of their Sylow subgroups. We saw in Exercise II.5.8 that if every Sylow subgroup of a finite group G is normal then G is the direct product of its Sylow subgroups. In this section we define nilpotent groups in terms of the behavior of certain associated groups. We see in Proposition II.7.5 that these are precisely the groups which are direct products of their Sylow subgroups.

Note. Recall that the center of a group G is $C(G) = \{a \in G \mid ax = xa \text{ for all } x \in G\}$. Notice that $C(G)$ is an abelian subgroup of G (by Problem I.2.11). By Corollary II.4.7 (see the proof) $C(G)$ is a normal subgroup of G. So consider the canonical homomorphism $\pi : G \to G/C(G)$ (where $\pi(g) = gC(G)$). Define $C_2(G) = \pi^{-1}[C(G/C(G))]$. By the proof of Theorem I.5.11, $C_2(G)$ is normal in G and contains $C(G)$. So we can define a chain of groups with $C_1(G) = C(G)$, and $C_i(G)$ is the inverse image of $C(G/C_{i-1}(G))$ under the canonical homomorphism mapping $G \to G/C_{i-1}(G)$. The "inverse image" part is necessary so that we actually get subgroups of G.

Definition II.7.1. The sequence of normal subgroups of G described above, $\langle e\rangle < C_1(G) < C_2(G) < \cdots$, is the *ascending central series* of G. Group G is *nilpotent* if $C_n(G) = G$ for some n.

Note. Since $G = C(G) = C_1(G)$ for abelian G, every abelian group is nilpotent.

Example 1. Let n be odd and consider D_n. The center of D_n is $C(D_n) = C_1(D_n) = \{e\}$ by Exercise I.6.12. So $D_n/\{e\} = \{d\{e\} \mid d \in D_n\} = \{\{d\} \mid d \in D_n\} \cong D_n$. Now $C(D_n/\{e\}) = \{\{e\}\}$ by Exercise I.6.2 and so $C_2(D_n) = \pi^{-1}(C(D_n/\{e\})) = \{e\}$. So this process is repeated so that $C_i(D_n) = \{e\}$ for all $i \in \mathbb{N}$, and the ascending central series for D_n (with n odd) is $\{e\} < \{e\} < \{e\} < \cdots$. Therefore D_n in not nilpotent when n is odd.

Example 2. Consider D_8 (a group of order 16). The center of D_8 is $C(D_8) = \{e, a^4\}$ by Exercise I.6.12 (in the solution, $C(D_n) = \{e, a^{n/2}\}$ for n even). Now $D_8/C(D_8) \cong D_4$ by Exercise I.6.A and so $C(D_8/C(D_8)) = \{\{e, a^4\}, a^2\{e, a^4\}\} = \{\{e, a^4\}, \{a^2, a^6\}\}$ (that is, the center of $D_8/C(D_8)$ is the identity coset and, since $D_8/C(D_8) \cong D_4$, the "rotation" a^2 of order 2). Then,

$$C_1(D_8) = \pi^{-1}(C(D_8/C(D_8))) = \pi^{-1}(\{e, a^4\}, \{a^2, a^6\}) = \{e, a^2, a^4, a^6\}$$

(since π maps elements of D_8 to cosets, π^{-1} maps cosets to their elements). Next, $D_8/C_1(D_8)$ is, by Lagrange's Theorem (Corollary I.4.6) of order 4 and so must be abelian. Hence

$$C(D_8/C_1(D_8)) = D_8/C_1(D_8)$$

$$= \{\{e, a^2, a^4, a^6\}, \{a, a^3, a^5, a^7\}, \{b, ba^2, ba^4, ba^6\}, \{ba, ba^3, ba^5, ba^7\}\}.$$

So $C_2(D_8) = \pi^{-1}(C(D_8/C_1(D_8)) = D_8$ and the ascending central series is $\{e\} < \{e, a^4\} < \{e, a^2, a^4, a^6\} < D_8$. So D_8 is nilpotent. ∎

Note. We now prove several properties of nilpotent groups.

Proposition II.7.2. Every finite p-group is nilpotent.

Proposition II.7.3. The direct product of a finite number of nilpotent groups is nilpotent.

Lemma II.7.4. If H is a proper subgroup of a nilpotent group G, then H is a proper subgroup of its normalizer $N_G(H)$.

Note. The following result classifies finite nilpotent groups in terms of their relationship to their Sylow p-subgroups.

Proposition II.7.5. A finite group is nilpotent if and only if it is (isomorphic to) the direct product of its Sylow subgroups.

Note. The following gives us a special case in which the converse of Lagrange's Theorem holds.

Corollary II.7.6. If G is a finite nilpotent group and m divides $|G|$, then G has a subgroup of order m.

Definition II.7.7. Let G be a group. The subgroup of G generated by the set $\{aba^{-1}b^{-1} \mid a, b \in G\}$ is called the *commutator subgroup* of G and denoted G'.

Note. Notice that G is abelian if and only if $G' = \{e\}$. Hungerford says (page 102) "In a sense, G' provides a measure of how much G differs from an abelian group." Of course, "in a sense" is a great, vague disclaimer! The following result somewhat elaborates on this.

Theorem II.7.8. If G is a group, then the commutator subgroup G' is a normal subgroup of G and G/G' is abelian. If N is a normal subgroup of G, then G/N is abelian if and only if N contains G'.

Note. Fraleigh (page 150 of the 7th edition) calls G/G' an "abelianized version" of G. Notice that it is the "largest" abelian quotient group of G since $N = G'$ is the "smallest" normal subgroup of G for which G/N is abelian, by Theorem II.7.8.

Example. We can use Theorem II.7.8 to find commutator subgroups. The Cayley table for S_3 is (in the notation of Fraleigh):

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_3	μ_1	μ_2
ρ_2	ρ_2	ρ_0	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	ρ_0	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	ρ_0	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	ρ_0

We see that S_3 has subgroup $N = \{\rho_0, \rho_1, \rho_2\}$ which is normal by Exercise I.5.1 since $[S_3 : N] = 2$. Now $S_3/N \cong \mathbb{Z}_2$ is abelian, so by Theorem II.7.8, N contains S_3'. We now show that ρ_0, ρ_1, ρ_2 are each commutators (in the notation of Exercise II.7.2 we denote the commutator of x and y as $[x, y] = xyx^{-1}y^{-1}$):

$$[\rho_0, \rho_0] = \rho_0\rho_0\rho_0^{-1}\rho_0^{-1} = \rho_0,$$
$$[\mu_1, \mu_3] = \mu_1\mu_3\mu_1^{-1}\mu_3^{-1} = \mu_1\mu_3\mu_1\mu_3 = \rho_2\rho_2 = \rho_1, \text{ and}$$
$$[\mu_1, \mu_2] = \mu_1\mu_2\mu_1^{-1}\mu_2^{-1} = \mu_1\mu_2\mu_1\mu_2 = \rho_1\rho_1 = \rho_2.$$

So $S_3' = N$. Since N is abelian then $N' = \{\rho_0\}$. So the sequence of derived subgroups of S_3 are: $S_3 > N > \{\rho_0\}$. That is, $S_3^{(2)} = \{\rho_0\}$. This means that S_3 is "solvable" as defined next.

Note. We now turn our attention to solvable groups. Solvable groups play a major role in showing the insolvability of the quintic (in Section V.9).

Definition. Let G be a group and let $G^{(1)} = G'$ be the commutator subgroup of G. For $i \geq 1$ define $G^{(i)} = (G^{(i-1)})'$ (that is, $G^{(i)}$ is the commutator subgroup of $G^{(i-1)}$). $G^{(i)}$ is the i^{th} *derived subgroup of* G.

Note. The derived subgroups of G produce a subgroup chain $G > G^{(1)} > G^{(2)} > \cdots$. By Theorem II.7.8 we know that each of these subgroup inclusions is in fact actually a normal subgroup inclusion. Each $G^{(i)}$ is a normal subgroup of G (and hence of each earlier subgroup in the chain) by Exercise II.7.13.

Definition II.7.9. A group G is said to be *solvable* if $G^{(n)} = \langle e \rangle$ for some n.

Note. Notice the similarity between the definition of nilpotent group and solvable group. Notice also that every abelian group G is solvable since, for such a group, $G^{(1)} = G' = \{e\}$.

Note. If you dealt with solvable groups in your senior level algebra class (Introduction to Modern Algebra, MATH 4137/5137) then this is likely *not* the definition of "solvable" which you encountered. In Fraliegh's 8th edition of *A First Course in Abstract Algebra*, a solvable group is defined as:

Definition 35.18. A group G is solvable if it has a composition series $\{H_i\}$ such that all factor groups H_{i+1}/H_i are abelian.

Here we see that solvability is ultimately related to the fact that *something* is abelian. This commutivity property is used by Niels Henrik Abel in his proof of the insolvability of the quintic (called "Abel's Theorem" in Hungerford's Proposition V.9.8). *This* is why he is commemorated by the term "abelian group" for a group with a commutative binary operation. In the next section (namely, in Theorem II.8.5) we will show that Hungerford's and Fraleigh's definitions of *solvable group* are equivalent.

Proposition II.7.10. Every nilpotent group is solvable.

Note. The converse of Proposition II.7.10 does not hold, since S_3 and S_4 are solvable but not nilpotent, as shown in Exercise II.7.10.

Theorem II.7.11.

(i) Every subgroup and every homomorphic image of a solvable group is solvable.

(ii) If N is a normal subgroup of a group G such that N and G/N are solvable, then G is solvable.

Note. The following is instrumental in the proof of the Insolvability of the Quintic. Notice that the proof includes the fact that A_n is not solvable for $n \geq 5$.

Corollary II.7.12. If $n \geq 5$, then the symmetric group S_n is not solvable.

Proof. ASSUME S_n is solvable for $n \geq 5$. Then, by Theorem II.7.11, subgroup A_n is solvable. Since A_n is nonabelian, then the commutator subgroup $A_n' \neq \{\iota\}$ (the trivial group) because a group G is abelian if and only if $G' = \{e\}$—see the note after the definition of commutator subgroup. By Theorem II.7.8, A_n' is normal in A_n. By Theorem I.6.10, A_n is simple for $n \geq 5$. So, by the definition of simple group, it must be that $A_n' = A_n$. But then the chain of derived subgroups $A^{(i)}$ consists only of copies of group A_n and does not terminate at $\{\iota\}$, implying that A_n is not solvable, a CONTRADICTION. So the assumption that S_n is solvable is false and, in fact, S_n is not solvable for $n \geq 5$. ∎

Note. The remainder of this section is not necessary for what follows (our main goals are an algebraic proof of the Fundamental Theorem of Algebra [see the appendix to Section V.3] and a proof of the Insolvability of the Quintic [see Abel's Theorem, Proposition V.9.8—we'll accomplish these goals in Modern Algebra 2 [MATH 5420]). However, we include the statement of a result which is a Sylow-type result for finite solvable groups.

Definition. A subgroup H of group G is a *characteristic subgroup* if $f(H) < H$ for every automorphism $f : G \to G$. Subgroup H is *fully invariant* if $f(H) < H$ for every endomorphism (that is, homomorphism from G to G) $f : G \to G$.

Note. Of course, a fully invariant subgroup is also characteristic (since an automorphism is an example of an endomorphism). Also, every characteristic subgroup is normal (since conjugation of G by an element of G is an automorphism of G and so the conjugation of H will be a subgroup of H if H is a characteristic subgroup and so H is normal by Theorem I.5.1(iv)).

Definition. A *minimal normal subgroup* of a group G is a nontrivial normal subgroup that contains no proper subgroup which is normal in G.

Lemma II.7.13. let N be a normal subgroup of a finite group G and H any subgroup of G.

(i) If H is a characteristic subgroup of N, then H is normal in G.

(ii) Every normal Sylow p-subgroup of G is fully invariant.

(iii) If G is solvable and N is a minimal normal subgroup, then N is an abelian p-group for some prime p.

Note. The following result is "Sylow-like" in that it gives subgroups of certain orders and shows that certain such subgroups are conjugate.

Proposition II.7.14. Let G be a finite solvable group of order mn, with $\gcd(m,n) = (m,n) = 1$. Then

(i) G contains a subgroup of order m;

(ii) any two subgroups of G of order m are conjugate;

(iii) any subgroup of G of order k, where $k \mid m$, is contained is a subgroup of order m.

Note. We omit the proof of Proposition II.7.14. It is about one and a half pages long and by now you know how brief Hungerford can be on details!

Note. Phillip Hall (1904–1982) proved Proposition II.7.14 in "A Note on Soluble Groups" *Journal of the London Mathematical Society*, **3** (1928), 98–105. If group G is of order mn and H is a subgroup of G of order m where $\gcd(m, n) = (m, n) = 1$, then H is called a *Hall subgroup* of G in his honor. For more information on him, see "Philip Hall. 11 April 1904 – 30 December 1982" *Biographical Memoirs of Fellows of the Royal Society* **30** (1984), 251–279. This is online at:

`http://rsbm.royalsocietypublishing.org/content/30/250`

(accessed 10/24/2014).

Note. If parameter m in Proposition II.7.14 is a power of a prime, then (i) implies the existence of a Sylow p-subgroup of G (from the First Sylow Theorem [Theorem II.5.7]). Part (ii) then implies that any two Sylow p-subgroups are conjugate (from the Second Sylow Theorem [Theorem II.5.9]).

Note. Phillip Hall has also proved the converse of Proposition II.7.14(i). Namely:
Theorem. If G is a finite group such that whenever $|G| = mn$ with $\gcd(m, n) = (m, n) = 1$ we have that G has a subgroup of order m, then group G is solvable.
Hungerford declares this result "beyond the scope of this book" (!) and references Marshall Hall's *The Theory of Groups* (1959).

Note. A prominent figure in the history of group theory is William Burnside (1852–1927). His famous book, *Theory of Groups of Finite Order* was first published in 1897 by Cambridge University Press. A second edition was published in 1911 and is still in print with Dover Publications and available through GoogleBooks (`http://books.google.com/books?id=rGMGAQAAIAAJ`, accessed 10/24/2014). He conjectured that every finite group of odd order is solvable. This was proved in 1963 by Walter Feit and John Thompson. The paper was 255 pages long and filled an entire issue of the *Proceedings of the London Mathematical Society*. The specific reference for the paper is "Solvability of Groups of Odd Order," *Proc. Lond. Math. Soc.*, **13** (1960), 775–1029. For more details, see my supplemental notes on "Finite Simple Groups" online at

`http://faculty.etsu.edu/gardnerr/4127/notes/Simple-Groups.pdf`.

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