Section II.8. Normal and Subnormal Series

Note. In this section, two more series of a group are introduced. These will be useful in the Insolvability of the Quintic. In addition, the Jordan-Hölder Theorem will be stated and proved. This result illustrates the usefulness of a study of finite simple groups.

Definition II.8.1. A *subnormal series* of a group G is a chain of subgroups $G = G_0 > G_1 > \cdots > G_n$ such that G_{i+1} is normal in G_i for $0 \le i < n$. The *factors* of the series are the quotient groups G_i/G_{i+1}. The *length* of the series is the number of strict inclusions (or equivalently, the number of nonidentity factors). A subnormal series such that G_i is normal in G for all i is a *normal series*.

Note. Of course every normal series is a subnormal series, but a subnormal series may not be normal (see Exercise I.5.10).

Note. We can have $K \triangleleft H$ and $H \triangleleft G$, but this does not necessarily imply that K is a normal subgroup of G. In Exercise II.7.9 it is shown that the commutator subgroup of $G = A_4$ is $H = A_4' \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In fact (with the usual notation for the permutation group S_4), $A_4' = \{(1)(2)(3)(4),\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\}$. By Theorem II.7.8, $A_4' \triangleleft A_4$. Now $K = \{(1)(2)(3)(4),\ (1,2)(3,4)\}$ is a normal subgroup of A_4' (by Exercise I.5.1), but K is not a normal subgroup of $G = A_4$ since for $(2,3)(1,3) = (1,2,3) \in A_4$ we have $(1,2,3)((1,2)(3,4))(1,2,3)^{-1} = (1,2,3)(1,2)(3,4)(1,3,2) = (1,3)(2,4) \notin K$. That is, $K \triangleleft H$ and $H \triangleleft G$, but K is not a normal subgroup of G. That is, the relation "is a normal subgroup" is not transitive.

Example. The derived series of commutator subgroups $G > G^{(1)} > G^{(2)} > \cdots G^{(n)}$ is a normal series, since each $G^{(i)}$ is normal in G by Exercise II.7.13. If G is nilpotent then the ascending central series $C_1(G) < C_2(G) < \cdots < C_n(G) = G$ is a normal series for G since each $C_i(G)$ is normal in G.

Definition II.8.2. Let $G = G_0 > G_1 > \cdots > G_n$ be a subnormal series. A *one-step refinement* of this series is any series of the form $G = G_0 > G_1 > \cdots > G_i > N > G_{i+1} > \cdots > G_n$ or $G = G_0 > G_1 > \cdots > G_n > N$, where N is a normal subgroup of G_i and (if $i < n$) G_{i+1} is normal in N. A *refinement* of a subnormal series S is any subnormal series obtained from S by a finite sequence of one-step refinements. A refinement of S is said to be *proper* if its length is larger then the length of S.

Definition II.8.3. A subnormal series $G = G_0 > G_1 > \cdots > G_n = \{e\}$ is a *composition series* if each factor G_i/G_{i+1} is simple. A subnormal series $G = G_0 > G_1 > \cdots > G_n = \{e\}$ is a *solvable series* if each factor is abelian.

Definition. $N \neq G$ is a *maximal normal subgroup* of G if there is no normal subgroup $M \neq G$ of G with $N \lhd M$ and $N \neq M$.

Note A. By Corollary I.5.12, if N is a normal subgroup of a group G, then every normal subgroup of G/N is of the form H/N where H is a normal subgroup of G which contains N. So when $G \neq N$, G/N is simple if and only if N is a maximal normal subgroup of G.

Theorem II.8.4.

(i) Every finite group G has a composition series.

(ii) Every refinement of a solvable series is a solvable series.

(iii) A subnormal series is a composition series if and only if it has no proper refinements.

Note. The following result resolves the difference between Fraleigh and Hungerford's definition of solvable groups.

Theorem II.8.5. A group G is solvable if and only if it has a solvable series.

Example. The dihedral group D_n is solvable. Let a be a generator of a subgroup of order n (say $a = \rho_1$, a "fundamental rotation" which generates the cyclic subgroup of D_n of order n consisting only of the rotations). Then we have the series $D_n > \langle a \rangle > \{e\}$ and $D_n/\langle a \rangle \cong \mathbb{Z}_2$ is abelian, and $\langle a \rangle/\{e\} \cong \mathbb{Z}_n$ is abelian. So this is a solvable series and by Theorem II.8.5, D_n is solvable.

Example. Let $|G| = pq$ where p and q are prime and, say, $p > q$. Then G has a normal subgroup of order p (and index q) by Corollary II.4.10. The subgroup is of prime order and so is cyclic, say it is $\langle a \rangle \cong \mathbb{Z}_p$. Then $G/\langle a \rangle$ is of order q and so is an abelian group (isomorphic to $\mathbb{Z}_q$) and $\langle a \rangle/\{e\} \cong \mathbb{Z}_p$ is abelian. So group G has a solvable series $G > \langle a \rangle > \{e\}$ and so by Theorem II.8.5 group G is solvable. This example with its factors $\mathbb{Z}_p$ and $\mathbb{Z}_q$ foreshadows the following result.

Proposition II.8.6. A finite group G is solvable if and only if G has a composition series whose factors are cyclic and of prime order.

Note. Since we can often refine a subnormal series of a given group, then we see that a group may have several different subnormal (or solvable) series. A group may also have different composition series, as shown in Exercise II.8.1. However, there is a type of uniqueness in terms of an equivalence, as given in the following definition.

Definition II.8.7. Two subnormal series S and T of a group G are *equivalent* if there is a one-to-one correspondence between the nontrivial factors of S and the nontrivial factors of T such that corresponding factors are isomorphic groups.

Fraleigh's Example 35.7. The two series $\{\overline{0}\} < \langle\overline{5}\rangle < \mathbb{Z}_{15}$ and $\{\overline{0}\} < \langle\overline{3}\rangle < \mathbb{Z}_{15}$ are equivalent normal series since the set of factor groups for $\{\overline{0}\} < \langle\overline{5}\rangle < \mathbb{Z}_{15}$ is $\{\mathbb{Z}_{15}/\langle\overline{5}\rangle \cong \mathbb{Z}_5, \langle\overline{5}\rangle/\langle\overline{0}\rangle \cong \mathbb{Z}_3\}$ and the set of factor groups for $\{\overline{0}\} < \langle\overline{3}\rangle < \mathbb{Z}_{15}$ is $\{\mathbb{Z}_{15}/\langle\overline{3}\rangle \cong \mathbb{Z}_3, \langle\overline{3}\rangle/\langle\overline{0}\rangle \cong \mathbb{Z}_5\}$.

Note. Two subnormal series of a given group do not have to have the same number of terms in order to be equivalent, but they do have to have the same "length" (that is, number of nontrivial factors).

Lemma II.8.8. If S is a composition series of a group G, then any refinement of S is equivalent to S.

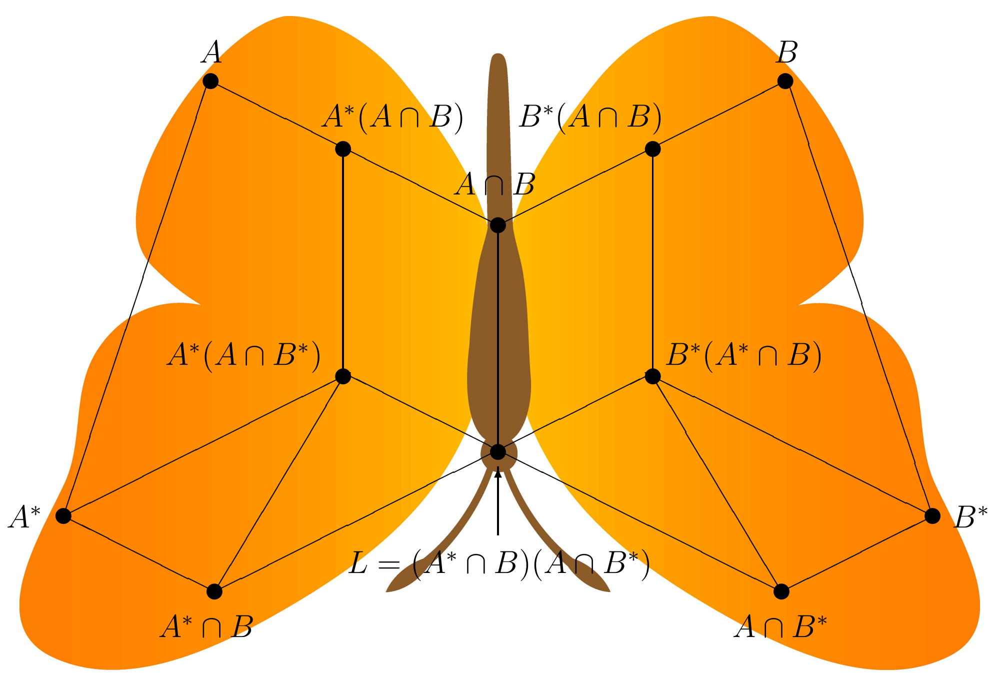
Note. The following result, the Zassenhaus Lemma or Butterfly Lemma is important in the role it plays in the proof of Schrier's Theorem and ultimately in the proof of the Jordan-Hölder Theorem. The result is called the "Butterfly Lemma" based on the shape of the subgroup diagram (this name is mentioned by Fraleigh [see page 313 of the 8th edition] but is not mentioned by Hungerford). This diagram appears on the back of the first t-shirts for the ETSU Abstract Algebra Club!

Lemma II.8.9. Zassenhaus' Lemma/The Butterfly Lemma.
Let A^*, A, B^*, B be subgroups of a group G such that A^* is normal in A and B^* is normal in B.

(i) $A^*(A \cap B^*)$ is a normal subgroup of $A^*(A \cap B)$.

(ii) $B^*(A^* \cap B)$ is a normal subgroup of $B^*(A \cap B)$.

(iii) $A^*(A \cap B)/A^*(A \cap B^*) \cong B^*(A \cap B)/B^*(A^* \cap B)$.



Theorem II.8.10. Schrier's Theorem.

Any two subnormal series of a group G have subnormal refinements that are equivalent. Any two normal series of a group G have normal refinements that are equivalent.

Fraleigh's Example 35.8. Consider the two normal series of $\mathbb{Z}$: (1) $\{\overline{0}\} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z}$ and (2) $\{\overline{0}\} < 9\mathbb{Z} < \mathbb{Z}$. Consider the refinement of (1) $\{\overline{0}\} < 72\mathbb{Z} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z}$ and the refinement of (2) $\{\overline{0}\} < 72\mathbb{Z} < 18\mathbb{Z} < 9\mathbb{Z} < \mathbb{Z}$. The four factor groups for both refinements are

$$72\mathbb{Z}/\{\overline{0}\} \cong 72\mathbb{Z},\ 8\mathbb{Z}/72\mathbb{Z} \cong \mathbb{Z}/9\mathbb{Z} \cong \mathbb{Z}_9,$$

$$4\mathbb{Z}/8\mathbb{Z} \cong 9\mathbb{Z}/18\mathbb{Z} \cong \mathbb{Z}_2,\ \mathbb{Z}/4\mathbb{Z} \cong 18\mathbb{Z}/72\mathbb{Z} \cong \mathbb{Z}_4.$$

Notice the factor groups are the same, although they appear in different orders. So there is a one to one correspondence between the factor groups $\{H_{i+1}/H_i\}$ and $\{K_{j+1}/K_j\}$. That is, the refinements are equivalent.

Example. Two composition series for $\mathbb{Z}_6$ are $\{\overline{0}\} < \{\overline{0}, \overline{2}, \overline{4}\} < \mathbb{Z}_6$ and $\{\overline{0}\} < \{\overline{0}, \overline{3}\} < \mathbb{Z}_6$. However these two series are equivalent since both have associated factor groups (isomorphic to) $\mathbb{Z}_2$ and $\mathbb{Z}_3$. This is no coincidence, as shown in the following.

Theorem II.8.11. Jordan-Hölder Theorem.

Any two composition series of a group G are equivalent. Therefore every group having a composition series determines a unique list of simple groups.

Note. Every finite group has a composition series by Theorem II.8.4(i). So the Jordan-Hölder Theorem implies that every finite group is associated with a (finite) list of simple groups. This fact lead to the 30 plus year exploration of finite simple groups. Daniel Gorenstein in his 1982 *Finite Simple Groups: An Introduction to Their Classification* (NY: Plenum Press, 1982) claims "In February 1981, the classification of the finite simple groups...was completed, representing one of the most remarkable achievements in the history of mathematics. Involving the combined efforts of several hundred mathematicians from around the world over a period of 30 years, the full proof covered something between 5,000 and 10,000 journal pages, spread over 300 to 500 individual papers." One of the papers is the 255 page paper by Walter Feit and John Thompson mentioned in the last section titled "Solvability of Groups of Odd Order" (*Proceedings of the London Mathematical Society*, **13** (1960), 775–1029). For more details on this project, see my handout for Introduction to Modern Algebra (MATH 4127/5127) on simple groups:

`http://faculty.etsu.edu/gardnerr/4127/notes/Simple-Groups.pdf`

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