Section III.4. Rings of Quotients and Localization

Note. Taking our lead from the rational numbers, in this section we define a ring of quotients (or "ring of fractions") for a given commutative ring R. The development mimics the way we might define ring $\mathbb{Q}$ in terms of ring $\mathbb{Z}$ (and involves the idea of equivalence classes in the since of "reducing" a fraction: $1/2 = 2/4 = 3/6 = \cdots$). Similar material is covered in Introduction to Modern Algebra (MATH 4127/5127) in Section IV.21. The Field of Quotients of an Integral Domain. At the end of this section, we consider "localizations" of a prime ideal (a topic which is referenced only occasionally in what follows and may be considered optional).

Definition III.4.1. A nonempty set S of a ring R is *multiplicative* provided that $a, b \in S$ implies $ab \in S$ (that is, S is closed under multiplication).

Example. In ring $\mathbb{R}$, sets $\mathbb{Q}$, $\mathbb{Z}$, $2\mathbb{Z}$, $\mathbb{Q}^+$, and $\mathbb{Z}^+$ are multiplicative. More generally, the set S of all elements in a nonzero ring with identity that are not zero divisors is multiplicative. The set of units in any ring with identity is a multiplicative set.

Note. In $\mathbb{Q}$, we have "multiple representations" of the same elements when considering quotients of integers: $a/b = c/d$ if and only if $ad = bc$. More precisely, consider $\mathbb{Z}$ as a ring and $S = \mathbb{Z}^+$ (the positive integers). We define a relation on set $\mathbb{Z} \times S$ as: $(a, b) \sim (c, d)$ if and only if $ad - bc = 0$. It is easily shown that this is an equivalence relation. We then define $\mathbb{Q}$ as the set of equivalence classes of $\mathbb{Z} \times S$ under this equivalence relation. We denote the equivalence class of (a, b) as "a/b." Addition and multiplication are defined as usual (once one verifies that defining them using representatives of the equivalence classes results in well-defined operations).

Note. We generalize this idea by starting with a commutative ring R. For S a multiplicative set, we define a commutative ring $S^{-1}R$ with an identity, and a homomorphism $\varphi_S : R \to S^{-1}R$ (in Theorems III.4.2 and III.4.3(i)). If S is the set of all nonzero elements in an integral domain R, then $S^{-1}R$ is a field (shown in Theorem III.4.3(iii)). The notation "$S^{-1}R$" is used for this commutative ring because we will create "quotients" with numerators from R and denominators from S. The proof of the following is straightforward and left as Exercise III.4.A.

Theorem III.4.2. Let S be a multiplicative subset of a commutative ring R. The relationship defined on set $R \times S$ by

$$(r, s) \sim (r', s') \text{ if and only if } s_1(rs' - r's) = 0 \text{ for some } s_1 \in S$$

is an equivalence relation. Furthermore if R has no zero divisors and $0 \notin S$ then $(r, s) \sim (r', s')$ if and only if $rs' - r's = 0$.

Note III.4.A. We denote the equivalence class $(r, s) \in R \times S$ as r/s. The set of all equivalence classes under $\sim$ is denoted $S^{-1}R$. It is straightforward to verify that:

(i) $r/s = r'/s'$ if and only if $s_1(rs' - r's) = 0$ for some $s_1 \in S$;

(ii) $(tr)/(ts) = r/s$ for all $r \in R$ and $s, t \in S$; and

(iii) If $0 \in S$, then $S^{-1}R$ consists of a single equivalence class.

Note III.4.B. By Theorem III.4.2, we have $(r, s) \sim (r', s')$ if and only if $s_1(rs' - r's) = 0$ for same $s_1 \in S$. So if $0 \in S$ then, with $s_1 = 0$, we have $(r, s) \sim (r', s')$ for all $r, r' \in R$ and $s, s' \in S$. That is, there is only one equivalence class under $\sim$ if we allow $0 \in S$. Think of S as the "denominators" in a quotient. Here we need not completely disallow **"division by 0"** (the setting is general enough to absorb this as a possibility), but in what results we get only one equivalence class. This means that the ring of quotients we define below (in Theorem III.4.3(i)) is the trivial ring with only one element. So *that's* the price we pay, even in this general setting, of trying to "divide by 0." Notice this is more of a legal loophole, than an actual result of any mathematical interest; the trivial ring $\{0\}$ satisfies the definition of a commutative ring with additive identity 0 and multiplicative identity 0. If we exclude 0 from S then more interesting things happen (namely, we get an integral domain or, if S is big enough, a field).

Note. We now establish some of the properties of $S^{-1}R$. After this, we will address the existence of certain homomorphisms.

Theorem III.4.3. Let S be a multiplicative subset of a commutative ring R and let $S^{-1}R$ be the set of equivalence classes of $R \times S$ under the equivalence relation of Theorem III.4.2.

(i) $S^{-1}R$ is a commutative ring with identity, where addition and multiplication are defined by $r/s + r'/s' = (rs' + r's)/(ss')$ and $(r/s)(r'/s') = (rr')/(ss')$.

(ii) If R is a nonzero ring with no zero divisors and $0 \notin S$, then $S^{-1}R$ is an integral domain.

(iii) If R is a nonzero ring with no zero divisors and S is the set of all nonzero elements of R, then $S^{-1}R$ is a field.

Definition. The ring $S^{-1}R$ in Theorem III.4.3 is the *ring of quotients* or *ring of fractions* of R by set S. (This should not be confused with a "quotient ring.") When S is the set of all nonzero elements in an integral domain R, the field $S^{-1}R$ of Theorem III.4.3(iii) is the *quotient field of the integral domain* R (or the *field of quotients*).

Note. With $R = \mathbb{Z}$ and $S = \mathbb{Z} \setminus \{0\}$, the field of quotients $S^{-1}R$ is $\mathbb{Q}$.

Definition. Suppose R a nonzero commutative ring and S the set of all nonzero elements of R that are not zero divisors. If S is nonempty (as is the case if R is a ring with identity), then $S^{-1}R$ is the *complete ring of quotients* of ring R. It is also called the *total ring of quotients* or *total ring of fractions*.

Note III.4.C. In light of Theorem III.4.3(iii) and the previous definition, we can state: "If a nonzero ring R has no zero divisors, then the complete ring of quotients of R is a field."

Note III.4.D. Exercise III.4.1 states: "Determine the complete ring of quotients of the ring $\mathbb{Z}_n$ for each $n \geq 2$." In this exercise, it is to be shown that the complete ring of quotients of $\mathbb{Z}_n$, $n \geq 2$, is isomorphic to $\mathbb{Z}_n$. Also, for $n = p$ prime we have $S = \mathbb{Z}_n \setminus \{\overline{0}\}$ and $S^{-1}\mathbb{Z}_n = S^{-1}\mathbb{Z}_p \cong \mathbb{Z}_p$ is a field (in this case $\mathbb{Z}_p$ is an integral domain; see Corollary III.4.6). As an example, for $\mathbb{Z}_6$ we take $S = \{\overline{1}, \overline{5}\}$ (notice that S is a multiplicative set; powers of $\overline{5}$ are equal to either $\overline{1}$ or $\overline{5}$). Then $S^{-1}\mathbb{Z}_6$ includes $\overline{0}/\overline{1}$, $\overline{1}/\overline{1}$, $\overline{2}/\overline{1}$, $\overline{3}/\overline{1}$, $\overline{4}/\overline{1}$, $\overline{5}/\overline{1}$, $\overline{0}/\overline{5}$, $\overline{1}/\overline{5}$, $\overline{2}/\overline{5}$, $\overline{3}/\overline{5}$, $\overline{4}/\overline{5}$, and $\overline{5}/\overline{5}$. Now $\overline{0}/\overline{1} = \overline{0}/\overline{5}$, $\overline{1}/\overline{1} = \overline{5}/\overline{5}$, $\overline{2}/\overline{1} = \overline{4}/\overline{5}$, $\overline{3}/\overline{1} = \overline{3}/\overline{5}$, $\overline{4}/\overline{1} = \overline{2}/\overline{5}$, and $\overline{5}/\overline{1} = \overline{1}/\overline{5}$ (these can be verified by "cross multiplying"). So there are six equivalence classes in $S^{-1}R$. Since the equivalence classes form an additive group of order six, then $\mathbb{Z}_6 \cong S^{-1}\mathbb{Z}_6$. In fact, the isomorphism is given by mapping $\bar{r} \mapsto \bar{r}/\overline{1}$ for $r \in \{0, 1, 2, 3, 4, 5\}$.

Note III.4.E. In a ring, a zero divisor cannot have an inverse. If r is a zero divisor in ring R, then $rr' = 0$ for some $r' \in R$. IF r is a unit (in which case R must be a ring with identity) then $r^{-1}r = 1$ for some $r^{-1} \in R$. But then $r^{-1}(rr') = r^{-1}(0)$ or $(r^{-1}r)r' = 0$ or $(1)r' = 0$, and hence $r' = 0$. But then r is not a zero divisor after all! Notice that we have used associativity in establishing that $r' = 0$ here. In fact, there are algebraic structures in which zero divisors do have inverses. The Cayley-Dickson algebra of the Sedenions is an algebraic structure with two binary operations, addition and multiplication, where multiplication is neither commutative nor associative (so this is NOT a ring). There are zero divisors in the Sedenions which have multiplicative inverses; in fact, every nonzero Sedenion has an inverse. For details, see my supplemental notes for Introduction to Modern Algebra 2 (MATH 4137/5137) on Supplement. The Cayley-Dickson Construction and Nonassociative Algebras.

Note III.4.F. So what happens if we compute a ring of quotients and include a zero divisor in set S? Consider, for example, the ring $\mathbb{Z}_6$ and let $S = \{\bar{3}\}$. Notice that S is a multiplicative set. So $S^{-1}\mathbb{Z}_6$ includes $\bar{0}/\bar{3}$, $\bar{1}/\bar{3}$, $\bar{2}/\bar{3}$, $\bar{3}/\bar{3}$, $\bar{4}/\bar{3}$, and $\bar{5}/\bar{3}$. Now $\bar{0}/\bar{3} = \bar{2}/\bar{3} = \bar{4}/\bar{3}$ and $\bar{1}/\bar{3} = \bar{3}/\bar{3} = \bar{5}/\bar{3}$ (these can be verified by "cross multiplying"). Therefore, $S^{-1}\mathbb{Z}_6$ only has two elements and so is isomorphic to the ring $\mathbb{Z}_2$. Notice that we have "lost" zero divisor $\bar{3}$ (we have even lost $\mathbb{Z}_6$), but its image under the mapping $\bar{3} \mapsto (\bar{3} \times \bar{3})/\bar{3} = \bar{3}/\bar{3}$ (which we'll see soon in Theorem III.4.4) is a unit in $S^{-1}\mathbb{Z}_6$; in fact, the image of $\bar{3}$ is the multiplicative identity in $S^{-1}\mathbb{Z}_6$ so that it is its own inverse.

Note. We now turn our attention to homomorphisms. If $\varphi : \mathbb{Z} \to \mathbb{Q}$ is the map $n \mapsto n/1$ then φ is a monomorphism that embeds $\mathbb{Z}$ in $\mathbb{Q}$. Also, for each nonzero $n \in \mathbb{Z}$, $\varphi(n)$ is a unit in $\mathbb{Q}$. We now generalize this idea.

Theorem III.4.4. Let S be a multiplicative subset of a commutative ring R.

(i) The map $\varphi_S : R \to S^{-1}R$ given by $r \mapsto rs/s$ (for any $s \in S$) is a well-defined homomorphism of rings such that $\varphi_S(s)$ is a unit in $S^{-1}R$ for every $s \in S$.

(ii) If $0 \notin S$ and S contains no zero divisors, then φ_S is a monomorphism. In particular, any integral domain may be embedded in its quotient field.

(iii) If R has an identity and S consists of units, then φ_S is an isomorphism. In particular, the complete ring of quotients of a field F is isomorphic to F.

Note III.4.G. The embedding φ_S of an integral domain R in its quotient field allows us to identify R with its image under φ_S (notice that $\varphi_S : R \to \text{Im}(\varphi)S$ is an isomorphism). In this case, we have $1 \in S$ and we identify $r \in R$ with equivalent class $r/1 \in S^{-1}R$.

Note. In the next theorem, a ring of quotients of commutative ring R is determined (up to isomorphism) in terms of a certain homomorphism, Hungerford says that "rings of quotients may be completely characterized by a universal mapping property" (see page 144). The "up to isomorphism" part of the claim as the approach taken by David Dummit and Richard Foote in *Abstract Algebra*, 3rd edition (John Wiley and Sons, 2004); see their Theorem 15.36 and definition of "ring of fractions of R with respect to [multiplicative set] D," $D^{-1}R$. They also call $D^{-1}R$ the "localization of R at D."

Theorem III.4.5. Let S be a multiplicative subset of a commutative ring R and let T be any commutative ring with identity. If $f : R \to T$ is a homomorphism of rings such that $f(s)$ is a unit in T for all $s \in S$, then there exists a unique homomorphism of rings $\overline{f} : S^{-1}R \to T$ such that $\overline{f}\varphi_S = f$. The ring $S^{-1}R$ is completely determined (up to isomorphism) by this property.

Note. The corollary to Theorem III.4.5 gives a sense in which the "smallest" field containing an integral domain is the field of quotients of the integral domain, as shown next.

Corollary III.4.6. Let R be an integral domain considered as a subring of its quotient field F (see Theorem III.4.4(ii)). If E is a field and $f : R \to E$ is a monomorphism of rings, then there is a unique monomorphism of rings, then there is a unique monomorphism of fields $\overline{f} : F \to E$, such that $\overline{f}|_R = f$. In particular, any field E_1 containing R contains an isomorphic copy F_1 of F with $R \subset F_1 \subset E_1$.

Note. The material in Theorems III.4.7 to III.4.11 are not needed until Section V.6, "Dedekind Domains," so we temporarily postpone the presentation of these results (which contain the "localization" part of this section). First, we consider local rings and Theorem III.4.13.

Definition III.4.12. A *local ring* is a commutative ring with identity which has a unique maximal ideal.

Note. An example of a local ring is given in Exercise III.4.13. In that exercise, it is shown that the ring R consisting of all rational numbers with denominators not divisible by some fixed prime p is a local ring. The solution of this exercise requires Theorem 4.11(ii), which we cover below.

Note III.4.H. Recall that Theorem III.2.18 states that a nonzero ring R with identity always has a maximal ideal, and that every ideal in R (except R itself; remember that R itself is not a maximal ideal of R by definition) is contained in a maximal ideal. So in a local ring, every deal of R (except E itself) is contained in the unique maximal ideal. For example, with $R = \mathbb{Z}_{p^n}$ where p is prime and $n \geq 1$, the unique maximal ideal is the one generated by $\overline{p}$, $(\overline{p})$. We now give two other conditions that classify local rings.

Theorem III.4.13. If R is a commutative ring with identity then the following conditions are equivalent:

(i) R is a local ring;

(ii) all nonunits of R are contained in some ideal $M \neq R$;

(iii) the nonunits of R form an ideal.

Note. We now back up and consider the material relevant to Theorems III.4.7 to III.4.11 and localization.

Theorem III.4.7. Let S be a multiplicative subset of a commutative ring R.

(i) If I is an ideal in R, then $S^{-1}I = \{a/s \mid a \in I, x \in S\}$ is an ideal in $S^{-1}R$.

(ii) If J is another ideal in R, then $S^{-1}(I + J) = S^{-1}I + S^{-1}J$, $S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$, and $S^{-1}(I \cap J) = (S^{-1}I)(S^{-1}J)$.

Definition. The ideal $S^{-1}I$ in $S^{-1}R$ of Theorem III.4.7(i) is the *extension* of I in $S^{-1}R$.

Note. In the event that commutative ring R has an identity, we get the following condition under which $S^{-1}I = S^{-1}R$.

Theorem III.4.8. Let S be a multiplicative subset of a commutative ring R with identity and let I be an ideal of R. Then $S^{-1}I = S^{-1}R$ if and only if $S \cap I \neq \varnothing$.

Note/Definition. By Exercise III.2.13 (applied to homomorphism $f = \varphi_S$), we have that if J is an ideal in ring of quotients $S^{-1}R$, then the inverse image $\varphi_S^{-1}(J)$ is an ideal in R. The ideal $\varphi_S^{-1}(J)$ is the *contraction* of J in R. We characterize prime ideals of a ring of quotients, after the next lemma.

Lemma III.4.9. Let S be multiplicative subset of a commutative ring R with identity and let I be an ideal in R.

(i) $I \subset \varphi_S^{-1}(S^{-1}I)$.

(ii) If $I = \varphi_S^{-1}(J)$ for some ideal J in $S^{-1}R$, then $S^{-1}I = J$. That is, every ideal in $S^{-1}R$ is of the form $S^{-1}I$ for some ideal I in R.

(iii) If P is a prime ideal in R and $S \cap P = \varnothing$, then $S^{-1}P$ is a prime ideal in $S^{-1}R$ and $\varphi_S^{-1}(S^{-1}P) = P$.

Theorem III.4.10. Let S be a multiplicative subset of a commutative ring R with identity. Then there is a one-to-one correspondence between the set $\mathcal{U}$ of prime ideals of R which are disjoint from S and the set $\mathcal{V}$ of prime ideals of $S^{-1}R$, given by $P \mapsto S^{-1}P$.

Definition. Let R be a commutative ring with identity and P a prime ideal of R. Then $S = R - P$ is a multiplicative subset of R (by Theorem III.2.15, the product of two non-elements of a prime ideal cannot be in the prime idea). The ring of quotients $S^{-1}R$ is the *localization of* R *at* P, denoted R_P. If I is an ideal in R, then the ideal $S^{-1}I$ in $R_P = S^{-1}R$ is denoted I_P.

Note. With $S = R - P$, the ring of quotients $R_P = S^{-1}R$ includes inverses of all elements of R that are NOT in P. In the next theorem we consider prime ideals of R contained in P and set up a one-to-one correspondence between these prime ideals and the prime ideals of $R_P = S^{-1}R$ (similar to the one-to-one correspondence of Theorem II.4.10).

Theorem III.4.11. Let P be a prime ideal in a commutative ring R with identity, and let $S = R - P$.

(i) There is a one-to-one correspondence between the set of prime ideals of R which are contained in P and the set of prime ideals of $R_p = S^{-1}R$, given by $Q \mapsto Q_P = S^{-1}Q$;

(ii) the ideal $P_P = S^{-1}P$ in R_P is the unique maximal ideal of R_P.

Note. For commutative ring R with identity, prime ideal P of R, and $S = R - P$, the quotient ring $R_P = S^{-1}R$ (i.e., the localization of R at P) has a unique maximal ideal $P_P = S^{-1}P$ by Theorem III4.11(ii). Therefore, by Definition 4.12, R_P is a local ring.

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