Section III.4. Rings of Quotients and Localization

**Note.** Taking our lead from the rational numbers, in this section we define a ring of quotients (or “ring of fractions”) for a given commutative ring $R$. The development mimics the way we might define ring $\mathbb{Q}$ in terms of ring $\mathbb{Z}$ (and involves the idea of equivalence classes in the sense of “reducing” a fraction: $1/2 = 2/4 = 3/6 = \cdots$). This same material is covered in Introduction to Modern Algebra (MATH 4127/5127) in Section IV.21. The Field of Quotients of an Integral Domain. At the end of this section, we consider “localizations” of a prime ideal (a topic which is referenced only occasionally in what follows and may be considered optional).

**Definition III.4.1.** A nonempty set $S$ of a ring $R$ is multiplicative provided that $a, b \in S$ implies $ab \in S$ (that is, $S$ is closed under multiplication).

**Example.** In ring $\mathbb{R}$, sets $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Q}^+$, and $\mathbb{Z}^+$ are multiplicative. More generally, the set $S$ of all elements in a nonzero ring with identity that are not zero divisors is multiplicative. The set of units in any ring with identity is a multiplicative set.

**Note.** In $\mathbb{Q}$, we have “multiple representations” of the same elements when considering quotients of integers: $a/b = c/d$ if and only if $ab = bc$. More precisely, consider $\mathbb{A}$ as a ring and $S = \mathbb{Z}^+$ (the nonnegative integers). We define a relation on set $\mathbb{Z} \times S$ as: $(a, b) \sim (c, d)$ if and only if $ad - bc = 0$. It is easily shown that this is an equivalence relation. We then define $\mathbb{Q}$ as the set of equivalence classes...
of $\mathbb{Z} \times S$ under this equivalence relation. We denote the equivalence class of $(a, b)$ as “$a/b$.” Addition and multiplication are defined as usual (once one verifies that defining them using representatives of the equivalence classes results in well-defined operations).

**Notes.** We generalize this idea by starting with a commutative ring $R$. For $S$ a multiplicative ring, $S^{-1}R$ with an identity and a homomorphism $\varphi_S : R \to S^{-1}R$ (in Theorems III.4.2 and III.4.3(i)). If $S$ is the set of all nonzero elements in an integral domain $R$, then $S^{-1}R$ is a field (shown in Theorem III.4.3(iii)). The proof of the following is straightforward and left as Exercise III.4.A.

**Theorem III.4.2.** Let $S$ be a multiplicative subset of a commutative ring $R$. The relationship defined on set $R \times S$ by

$$(r, s) \sim (r', s') \text{ if and only if } s_1(rs' - r's) = 0 \text{ for some } s_1 \in S$$

in an equivalence relation. Furthermore if $R$ has no zero divisors and $0 \notin S$ then $(r, s) \sim (r', s') \text{ if and only if } rs' - r's = 0$.

**Note III.4.A.** We denote the equivalence class $(r, s) \in R \times S$ as $r/s$. The set of all equivalence classes under $\sim$ is denoted $S^{-1}R$. It is straightforward to verify that:

(i) $r/s = r'/s'$ if and only if $s_1(rs' - r's) = 0 \text{ for some } s_1 \in S$;

(ii) $(tr)/(ts) = r/s$ for all $r \in R$ and $s, t \in S$; and

(iii) If $0 \in S$, then $S^{-1}R$ consists of a single equivalence class.
Note. We now establish some of the properties of $S^{-1}R$. After this, we will address the existence of certain homomorphisms.

**Theorem III.4.3.** Let $S$ be a multiplicative subset of a commutative ring $R$ and let $S^{-1}R$ be the set of equivalence classes of $R \times S$ under the equivalence relation of Theorem III.4.2.

(i) $S^{-1}R$ is a commutative ring with identity, where addition and multiplication are defined by

\[ \frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'} \text{ and } \frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}. \]

(ii) If $R$ is a nonzero ring with no zero divisors and $0 \notin S$, then $S^{-1}R$ is an integral domain.

(iii) If $R$ is a nonzero ring with no zero divisors and $S$ is the set of all nonzero elements of $R$, then $S^{-1}R$ is a field.

**Definition.** The ring $S^{-1}R$ in Theorem III.4.3 is the ring of quotients or ring of fractions of $R$ by set $S$. When $S$ is the set of all nonzero elements in an integral domain $R$, the field $E^{-1}R$ of Theorem III.4.3(iii) is the quotient field of the integral domain $R$ (or the field of quotients).

Note. With $R = \mathbb{Z}$ and $X = \mathbb{Z} \setminus \{0\}$, the field of quotients $S^{-1}R$ is $\mathbb{Q}$.
Definition. For $R$ a nonzero commutative ring and $S$ the set of all nonzero elements of $R$ that are not zero divisors. If $S$ is nonempty (as is the case if $R$ is a ring with identity), then $S^{-1}R$ is the complete ring of quotients of ring $R$.

Note. We now turn our attention to homomorphisms. If $\varphi : \mathbb{Z} \to \mathbb{Q}$ is the map $n \mapsto n/1$ then $\varphi$ is a monomorphism that embeds $\mathbb{Z}$ in $\mathbb{Q}$. Also, for each nonzero $n \in \mathbb{Z}$, $\varphi(n)$ is a unit in $\mathbb{Q}$. We now generalize this idea.

Theorem III.4.4. Let $S$ be a multiplicative subset of a commutative ring $R$.

(i) The map $\varphi_S : R \to S^{-1}R$ given by $r \mapsto rs/s$ (for any $s \in S$) is a well-defined homomorphism of rings such that $\varphi_S(s)$ is a unit in $S^{-1}R$ for every $s \in S$.

(ii) If $0 \notin S$ and $S$ contains no zero divisors, then $\varphi_S$ is a monomorphism. In particular, any integral domain may be embedded in its quotient field.

(iii) If $R$ has an identity and $S$ consists of units, then $\varphi_S$ is an isomorphism. In particular, the complete ring of quotients of a field $F$ is isomorphic to $F$.

Revised: 7/27/2021