Section IV.1. Modules, Homomorphisms, and Exact Sequences

Note. In this section, we define a module (and vector space) and develop basic properties and definitions, such as homomorphisms, isomorphisms, submodules, products, sums, and exact sequences.

Definition IV.1.1. Let R be a ring. A *left R-module* is an additive abelian group A together with a function mapping $R \times A \to A$ (the image of (r, a) being denoted ra) such that for all $r, s \in R$ and $a, b \in A$:

(i) $r(a + b) = ra + rb$;

(ii) $(r + s)a = ra + sa$;

(iii) $r(sa) = (rs)a$.

If R has an identity 1_R and

(iv) $1_R a = a$ for all $a \in A$,

then A is a *unitary R-module.* If R is a division ring, then a unitary R-module is called a *left vector space.* Right R-modules are similarly defined by a function mapping $A \times R \to A$ with the obvious analogues.

Note. We will use the term "R-module" and adopt the notation of left R-modules, with corresponding results for right R-modules following similarly. With the notation of Definition IV.1.1, we will refer to "R-module A."

Note IV.1.A. Notice that an R-module is similar to a vector space where the scalars come from ring R and the vectors from abelian group A. However, Hungerford defines a "vector space" where the "scalars" come from a division ring (so all nonzero scalars have multiplicative inverses, but the scalars may not be commutative under multiplication). In analysis, you only study vector spaces over fields (usually either $\mathbb{R}$ or $\mathbb{C}$). In fact, Fraleigh's undergraduate text (*A First Course in Abstract Algebra*, 7th edition, Pearson 2002), Gallian's undergraduate text (*Contemporary Abstract Algebra*, 8th edition, Cengage Learning 2012), and David Dummit and Richard Foote's graduate level *Abstract Algebra*, 3rd edition (Wiley 2003), each define vector spaces over fields, so Hungerford's approach is maybe nonstandard.

Note IV.1.B. In R-module over A where 0_R is the additive identity in R, 0_A is the additive identity in A, $r \in R$, and $a \in A$ then we have

$$r0_A = 0_A \text{ and } 0_R a = 0_A.$$

Hungerford denotes all of these, as well as $0 \in \mathbb{Z}$ and the trivial module $\{0\}$, as "0." We also have for all $r \in R$, $n \in \mathbb{Z}$, and $a \in A$:

$$(-r)a = -(ra) = r(-a) \text{ and } n(ra) = r(na).$$

Example IV.1.A. Every additive abelian group G is a unitary $\mathbb{Z}$-module with na, where $n \in \mathbb{Z}$ and $a \in A$, defined as

$$na = \begin{cases} \underbrace{a + a + \cdots + a}_{n \text{ times}} & \text{if } n \geq 0 \\ \underbrace{(-a) + (-a) + \cdots + (-a)}_{|n| \text{ times}} & \text{if } n < 0. \end{cases}$$

Example. If S is a ring and R is a subring of S, then S is an R-module with ra defined as the product of r and a in S.

Example. Let R and S be rings and $\varphi : R \to S$ be a ring homomorphism. Then every S-module A can be made into an R-module by defining for each $x \in A$, rx as $\varphi(r)x$. The R-module structure of A is said to be given by *pullback along* φ.

Example. Let R be a ring with unity and $n \in \mathbb{N}$. Define $R^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in R\}$. Define

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_b + b_n)$$

and $\alpha(a_1, a_2, \ldots, a_n) = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)$ for all $\alpha \in R$. Then R^n is an R-module called the *free module of rank n over R.*

Definition IV.1.2. Let A and B be modules over a ring R. A function $f : A \to B$ is an *R-module homomorphism* provided that for all $a, c \in A$ and $r \in R$:

$$f(a + c) = f(a) + f(c) \text{ and } f(ra) = rf(a).$$

If R is a division ring, then an R-module homomorphism is a *linear transformation.* R-module monomorphism/epimorphism/isomorphism, are defined in the obvious way. The *kernel* of a homomorphism is $\mathrm{Ker}(f) = \{a \in A \mid f(a) = 0\}$. The *image* of f is $\{b \in B \mid b = f(a) \text{ for some } a \in A\}$.

Note IV.1.C. By Theorem I.2.3 (considering the R-module homomorphism f as a group homomorphism applied to the additive abelian groups A and B), we have the following two properties:

(i) f is an R-module monomorphism if and only if $\mathrm{Ker}(f) = \{0\}$;

(ii) $f : A \to B$ is an R-module isomorphism if and only if there is an R-module homomorphism $g : B \to A$ such that $g \circ f = gf = 1_A$ and $f \circ g = fg = 1_B$.

Example. If R is a ring then $R[x]$ is an R-module and $f : R[x] \to R[x]$ where $f(p(x)) \to xp(x)$ is an R-module homomorphism but not a ring homomorphism (because $f(p(x)q(x)) \neq f(p(x))f(q(x))$).

Note IV.1.D. For a given ring R, the class of all R-modules and R-module homomorphisms form a category. Similarly (and more important to the content of Section IV.2. Free Modules and Vector Spaces) for a given ring R with a unit, the class of all unitary R-modules and R-module homomorphisms form a category. This allows us to define ring epimorphisms and monomorphisms "strictly in categorical terms." That is, these mappings can be defined without reference to elements of the R-module, but instead in terms of the morphisms. This is to be shown in Exercise IV.1.2, for example, where R-module homomorphism f is shown to be a monomorphism if and only if every pair of R-module homomorphisms, g and h, such that $fg = fh$, we have $g = h$. Notice this only involves properties of morphisms (as just claimed, the morphisms are the R-module homomorphisms in the category of all R-modules).

Definition IV.1.3. Let R be a ring, A an R-module and B a nonempty subset of A. B is a *submodule* of A provided B is an additive subgroup of A and $rb \in B$ for all $r \in R$ and $b \in B$. A submodule of a vector space over a division ring is a *subspace*.

Example IV.1.B. If R is a ring and $f : A \to B$ is an R-module homomorphism, then $\mathrm{Ker}(f)$ is a submodule of A and $\mathrm{Im}(f)$ is a submodule of B. If C is any submodule of B then $f^{-1}(C) = \{a \in A \mid f(a) \in C\}$ is a submodule of A.

Example. For the R-module R^n defined above, R^m is a submodule of R^n for all $1 \le m \le n$.

Example. Let I be a left ideal of the ring R, A an R-module and S a nonempty subset of A. In Exercise IV.1.3(a) you are to show that $IS = \{\sum_{i=1}^n r_i a_i \mid r_i \in I, a_i \in S, n \in \mathbb{N}\}$ is a submodule of A.

Definition IV.1.4. If X is a subset of a module A over a ring R, then the intersection of all submodules of A containing X is the *submodule generated by* X (or "spanned" by X).

Definition. If X is finite and X generates module B, then B is *finitely generated.* If $|X| = 1$ then B is a cyclic module generated by $X = \{z\}$. If $\{B_i \mid i \in I\}$ is a family of submodules of A, then the submodule of A generated by $X = \cup_{i \in I} B_i$ is the *sum* of modules B_i, denoted $B_1 + B_2 + \cdots + B_n$ if set I is finite.

Theorem IV.1.5. Let R be a ring, A an R-module, X a subset of A, $\{B_i \mid i \in I\}$ a family of submodules of A and $a \in A$. Let $Ra = \{ra \mid r \in R\}$. Then the following hold:

(i) Ra is a submodule of A and the map $R \to Ra$ given by $r \mapsto ra$ is an R-module epimorphism.

(ii) The cyclic submodule C generated by a is $\{ra + na \mid r \in R, n \in \mathbb{Z}\}$. If R has an identity and C is unitary, then $C = Ra$.

(iii) The submodule D generated by X is

$$\left\{ \sum_{i=1}^{s} r_i a_i + \sum_{j=1}^{t} n_j b_j \;\middle|\; s, t \in \mathbb{N}; a_i, b_j \in X; r_i \in R; n_j \in \mathbb{Z} \right\}.$$

If R has an identity and A in unitary, then

$$D = RX = \left\{ \sum_{i=1}^{s} r_i a_i \;\middle|\; s \in \mathbb{N}; a_i \in X; r_i \in R \right\}.$$

(iv) The sum of the family $\{B_i \mid i \in I\}$ consists of all finite sums $b_{i_1} + b_{i_2} + \cdots + b_{i_n}$ with $b_{i_k} \in B_{i_k}$.

Note. The proof of Theorem IV.1.5 is straightforward and left as an exercise.

Theorem IV.1.6. Let B be a submodule A over a ring R. Then the quotient group A/B is an R-module with the action of R on A/B given by

$$r(a+B) = rB \text{ for all } r \in R, a \in A.$$

The map $\pi : A \to A/B$ given by $a \mapsto a + B$ is an R-module epimorphism with kernel B.

Definition. The map $\pi : A \to A/B$ in Theorem IV.1.6 is the *canonical epimorphism* (or *canonical projection*) of module A onto module A/B.

Note IV.1.E. The canonical epimorphism of modules closely resembles the canonical epimorphism of a group and a quotient group (see Section I.5. Normality, Quotient Groups, and Homomorphisms), so it is not surprising that many results which hold for quotient groups also hold in the module setting (the group homomorphisms just need to be confirmed to be module homomorphisms). The next four results (Theorem IV.1.7, Corollary IV.1.8, Theorem IV.1.9, and Theorem IV.1.10) correspond to the group results given in Theorems I.5.6 to I.5.12.

Theorem IV.1.7. If R is a ring and $f : A \to B$ is an R-module homomorphism and C is a submodule of $\text{Ker}(f)$, then there is a unique R-module homomorphism $\tilde{f} : A/C \to B$ such that $\tilde{f}(a + C) = f(a)$ for all $a \in A$; $\text{Im}(\tilde{f}) = \text{Ker}(f)/C$. $\tilde{f}$ is an R-module isomorphism if and only if f is an R-module epimorphism and $C = \text{Ker}(f)$. In particular, $A/\text{Ker}(f) \cong \text{Im}(f)$.

Corollary IV.1.8. If R is a ring and A' is a submodule of the R-module A and B' a submodule of the R-module B and $f : A \to B$ is an R-module homomorphism such that $f(A') \subset B'$, then f induces an R-module homomorphism $\overline{f} : A/A' \to B/B'$ given by $a + A' \mapsto f(a) + B'$. $\overline{f}$ is an R-module isomorphism if and only if $\mathrm{Im}(f) + B' = B$ and $f^{-1}(B') \subset A'$. In particular if f is an epimorphism such that $f(A') = B'$ and $\mathrm{Ker}(f) \subset A'$, then $\overline{f}$ is an R-module isomorphism.

Theorem IV.1.9. Let B and C be submodules of a module A over a ring R.

(i) There is an R-module isomorphism $B/(B \cap C) \cong (B + C)/C$;

(ii) if $C \subset B$, then B/C is a submodule of A/C, and there is an R-module isomorphism $(A/C)/(B/C) \cong A/B$.

Theorem IV.1.10. If R is a ring and B is a submodule of an R-module A, then there is a one-to-one correspondence between the set of all submodules of A containing B and the set of all submodules of A/B, given by $C \mapsto C/B$. hence every submodule of A/B is of the form C/B, where C is a submodule of A which contains B.

Theorem IV.1.11. Let R be a ring and $\{A_i \mid I\}$ a nonempty family of R-modules, $\prod_{i \in I} A_i$ the direct product of the abelian groups A_i, and $\sum_{i \in I} A_i$ the direct sum of the abelian groups A_i.

(i) $\prod_{i \in I} A_i$ is an R-module with the action of R given by $r\{a_i\} = \{ra_i\}$.

(ii) $\sum_{i\in I} A_i$ is a submodule of $\prod_{i\in I} A_i$.

(iii) For each $k \in I$, the canonical projection $\pi_k : \prod A_i \to A_k$ (Theorem I.8.1) is an R-module epimorphism.

(iv) For each $k \in I$, the canonical injection $\iota_k : A_k \to \sum A_i$ (Theorem I.8.4) is an R-module monomorphism.

Definition. The module $\prod_{i\in I} A_i$ of Theorem IV.1.11(i) is called the (external) *direct product* of the family of R-modules $\{A_i \mid i \in I\}$ and $\sum_{i\in I} A_i$ is the (external) *direct sum* of the set. If the index is finite, say $I = \{1, 2, \ldots, n\}$, then we claim that the direct product and direct sum coincide (recall that the direct product and direct sum of a finite collection of groups are distinguished only based on the notation used, multiplicative versus additive; see Definition I.8.3 in Section I.8. Direct Products and Direct Sums) and is written as $A_1 \oplus A_2 \oplus \cdots \oplus A_n$. The map $\pi_k : \prod_{i\in I} A_i \to A_k$ of Theorem IV.1.11(iii) is the *canonical projection*. The map $\iota_k : A_k \to \sum A_k$ of Theorem IV.1.11(iv) is the *canonical injection*.

Theorem IV.1.12. If R is a ring, $\{A_i \mid i \in I\}$ a family of R-modules, C an R-module, and $\{\varphi_i : C \to A_i \mid i \in I\}$ a family of R-module homomorphisms, then there is a unique R-module homomorphism $\varphi : C \to \prod_{i\in I} A_i$ such that $\pi_i\varphi = \varphi_i$ for all $i \in I$. $\prod_{i\in I} A_i$ is uniquely determined up to isomorphism by this property. In other words, $\prod_{i\in I} A_i$ is a product in the category of R-modules.

Theorem IV.1.13. If R is a ring, $\{A_i \mid i \in I\}$ a family of R-modules, D an R-module, and $\{\psi_i : A_i \to D \mid i \in I\}$ a family of R-module homomorphisms, then there is a unique R-module homomorphism $\psi : \sum_{i\in I} A_i \to D$ such that $\psi\iota_i = \psi_i$ for all $i \in I$. $\sum_{i\in I} A_i$ is uniquely determined up to isomorphism by this property. In other words, $\sum_{i\in I} A_i$ is a coproduct in the category of R-modules.

Note IV.1.F. Let A and B be R-modules. If $f : A \to B$ and $g : A \to B$ are R-module homomorphisms then we can define $f + g : A \to B$ where $f + g$ maps $a \mapsto f(a) + g(a)$. In Exercise IV.1.7, you are asked to show that the set of all R-module homomorphisms mapping $A \to B$, denoted $\mathrm{Hom}_R(A, B)$, is an abelian group under this addition.

Theorem IV.1.14. Let R be a ring and $A, A_1, A_2, \ldots, A_n$ R-modules. Then $A \cong A_i \oplus A_2 \oplus \cdots \oplus A_n$ if and only if for each $i = 1, 2, \ldots, n$ there are R-module homomorphisms $\pi_i : A \to A_i$ and $\iota_i : A_i \to A$ such that

(i) $\pi_i\iota_i = 1_{A_i}$ for $i = 1, 2, \ldots, n$;

(ii) $\pi_j\iota_i = 0$ for $i \neq j$;

(iii) $\iota_1\pi_1 + \iota_2\pi_2 + \cdots + \iota_n\pi_n = 1_A$.

Theorem IV.1.15. Let R be a ring and $\{A_i \mid i \in I\}$ a family of submodules of an R-module A such that

(i) A is the sum of the family $\{A_i \mid i \in I\}$;

(ii) for each $k \in I$, $A_k \cap A_k^* = 0$. where A_k^* is the sum of the family $\{A_i \mid i \neq k\}$.

Then there is an isomorphism $A \cong \sum_{i \in I} A_i$.

Note. We leave the proof of Theorem IV.1.15 as an exercise.

Definition. Let R be a ring and $\{A_i \mid i \in I\}$ a family of submodules of an R-module A. If A and $\{A_i\}$ satisfy (i) and (ii) of Theorem IV.1.15 then A is the (*internal*) *direct sum* of $\{A_i \mid i \in I\}$. This is denoted $A = \sum_{i \in I} A_i$ (but see the following note).

Note IV.1.G. The internal direct sum of Theorem IV.1.15 has a subtle difference from the external direct sum in Theorem IV.1.11. With A as the <u>internal</u> direct sum of A_i, each A_i is a submodule of A and <u>A is isomorphic</u> to the external direct sum $\sum_{i \in I} A_i$. With the <u>external</u> direct sum $\sum_{i \in I} A_i$, the A_i are not submodules of $\sum_{i \in I} A_i$ but instead their isomorphic images $\iota_i(A_i)$ are submodules of the external direct product (where the ι_i are the canonical injections). Hungerford says "this distinction if unimportant in practice" (see his page 175) so, like him, we adopt the notation $A = \sum_{i \in I} A_i$ to indicate a direct sum, internal or external.

Definition IV.1.16. A pair of module homomorphisms, $A \xrightarrow{f} B \xrightarrow{g} C$, is *exact* at B provided $\mathrm{Im}(f) = \mathrm{Ker}(g)$. A finite sequence of module homomorphisms, $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n$, is *exact* provided $\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$ for $i = 1, 2, \ldots, n-1$. An infinite sequence of module homomorphisms,

$$\cdots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \cdots$$

is *exact* provided $\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$ for all $i \in \mathbb{Z}$.

Example. For A and B modules, the sequences

$$\{0\} \longrightarrow A \xrightarrow{\iota} A \oplus B \xrightarrow{\pi_B} B \longrightarrow \{0\} \text{ and } \{0\} \longrightarrow B \xrightarrow{\iota} A \oplus B \xrightarrow{\pi_A} A \longrightarrow \{0\},$$

are exact, where there are unique homomorphisms mapping $\{0\}$ to an R-module (namely, $0 \mapsto 0_A$, say) and mapping A to $\{0\}$ (namely, $a \mapsto 0$ for all $a \in A$), ι is the canonical injection and π is the canonical projection.

Example. If C is a submodule of D, then the sequence

$$\{0\} \longrightarrow C \xrightarrow{i} D \xrightarrow{p} D/C \longrightarrow \{0\}$$

is exact where i is the inclusion map (see page 4) and p is the canonical epimorphism (see Theorem IV.1.6).

Definition. If $f : A \to B$ is a module homomorphism then $A/\mathrm{Ker}(f)$ is the *coimage* of f, denoted $\mathrm{Coim}(f)$. $B/\mathrm{Im}(f)$ is the *cokernel* of f denoted $\mathrm{Coker}(f)$.

Example. If $f : A \to B$ is a module homomorphism then the following sequences are exact:

$$\{0\} \to \mathrm{Ker}(f) \to A \to \mathrm{Coim}(f) \to \{0\}$$

$$\{0\} \to \mathrm{Im}(f) \to B \to \mathrm{Coker}(f) \to \{0\}$$

$$\{0\} \to \mathrm{Ker}(f) \to A \xrightarrow{f} \mathrm{Coker}(f) \to \{0\}$$

where the unlabeled mappings are inclusions (for the first two mappings on the left) and projections (for the last two mappings on the right). Notice that $\mathrm{Coim}(f) = A/\mathrm{Ker}(f)$ and $\mathrm{Coker}(f) = B/\mathrm{Im}(f)$, so the first projection is is the canonical epimorphism of Theorem IV.1.6.

Note IV.1.H. The sequence $\{0\} \to A \xrightarrow{f} B$ is an exact sequence if and only if f is a monomorphism (and so $\mathrm{Ker}(f) = \{0\}$, by Theorem I.2.3). Similarly $B \xrightarrow{g} C \to C = \{0\}$ is exact if and only if g is onto (that is, an epimorphism). If $A \xrightarrow{f} B \xrightarrow{g} C$ is exact then $gf = f \circ f = 0$ (the 0 function mapping $A \to \{0_C\}$). If $A \xrightarrow{f} B \xrightarrow{g} C \to \{0\}$ is exact, then

$$\begin{aligned}
\mathrm{Coker}(f) &= B/\mathrm{Im}(f) \text{ by definition of } \mathrm{Coker}(f) \\
&= B/\mathrm{Ker}(g) \text{ since } \mathrm{Im}(f) = \mathrm{Ker}(g) \text{ by definition of “exact”} \\
&= \mathrm{Coim}(g) \text{ by definition of } \mathrm{Coim}(g) \\
&\cong C \text{ since } g \text{ must be onto (an epimorphism).}
\end{aligned}$$

Definition. An exact sequence of the form $\{0\} \to A \xrightarrow{f} B \xrightarrow{g} C \to \{0\}$ is a *short exact sequence.*

Note IV.1.I. In the short exact sequence above, f must be a monomorphism and g must be an epimorphism Note IV.1.H. Hungerford comments that a short exact sequence is a way of presenting a submodule of B ($A \cong \mathrm{Im}(f) \subseteq B$ since f is one to one) and its quotient module $B/\mathrm{Im}(f)$ (which is isomorphic it $B/\mathrm{Ker}(g) \cong C$).

Note. Recall from Section 0.3. Functions, a diagram is commutative if we can follow two different directed paths through the diagram from object A to object B and if the compositions of the corresponding maps are equal.

Lemma IV.1.17. The Short Five Lemma.

Let R be a ring and

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd}$$

a commutative diagram of R-modules and R-module homomorphisms such that each row is a short exact sequence. Then

(i) if α and γ are monomorphisms then β is a monomorphism;

(ii) if α and γ are epimorphisms then β is an epimorphism;

(iii) if α and γ are isomorphisms then β is an isomorphism.

Note. Lemma IV.1.17 is called the "Short" Five Lemma because it deals with short exact sequences. A related result is given in Exercise IV.1.12:

The Five Lemma.

Let

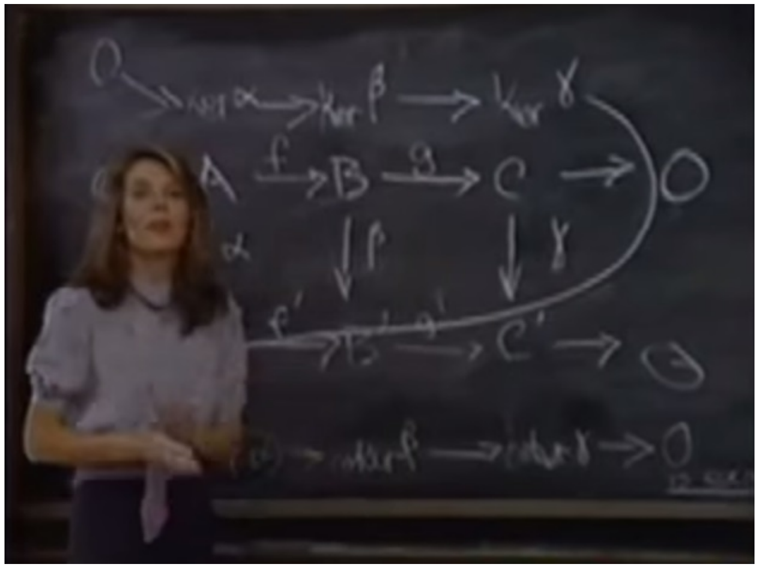
$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\alpha_1"] & A_2 \arrow[r] \arrow[d, "\alpha_2"] & A_3 \arrow[r] \arrow[d, "\alpha_3"] & A_4 \arrow[r] \arrow[d, "\alpha_4"] & A_5 \arrow[d, "\alpha_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

be a commutative diagram of R-modules and R-module homomorphisms, with exact rows. Then

(a) if α_1 is an epimorphism and α_2, α_4 are monomorphisms then α_3 is a monomorphism;

(b) if α_5 is a monomorphism and α_2, α_4 are epimorphisms then α_3 is an epimorphism.

Note IV.1.J. Another result related to the Short Five Lemma had a bit of celebrity in the 1980 Rastar Films' *It's My Turn*, staring Jill Clayburgh and Michael Douglas. Clayburgh portrays an algebraist and there is a seen where she is explaining the *Snake Lemma*. The scene is available on YouTube at (accessed 12/17/2023). The Snake Lemma deals with the existence of an exact sequence including kernels and cokernals of the mappings in the Short Five Lemma. This is, to my knowledge, the only big-budget movie with a lead character who is an algebraist. The image below is from the YouTube website.



Jill Clayburgh as Dr. Kate Gunzinger, presenting the Snake Lemma

The Snake Lemma. Let R be a ring and

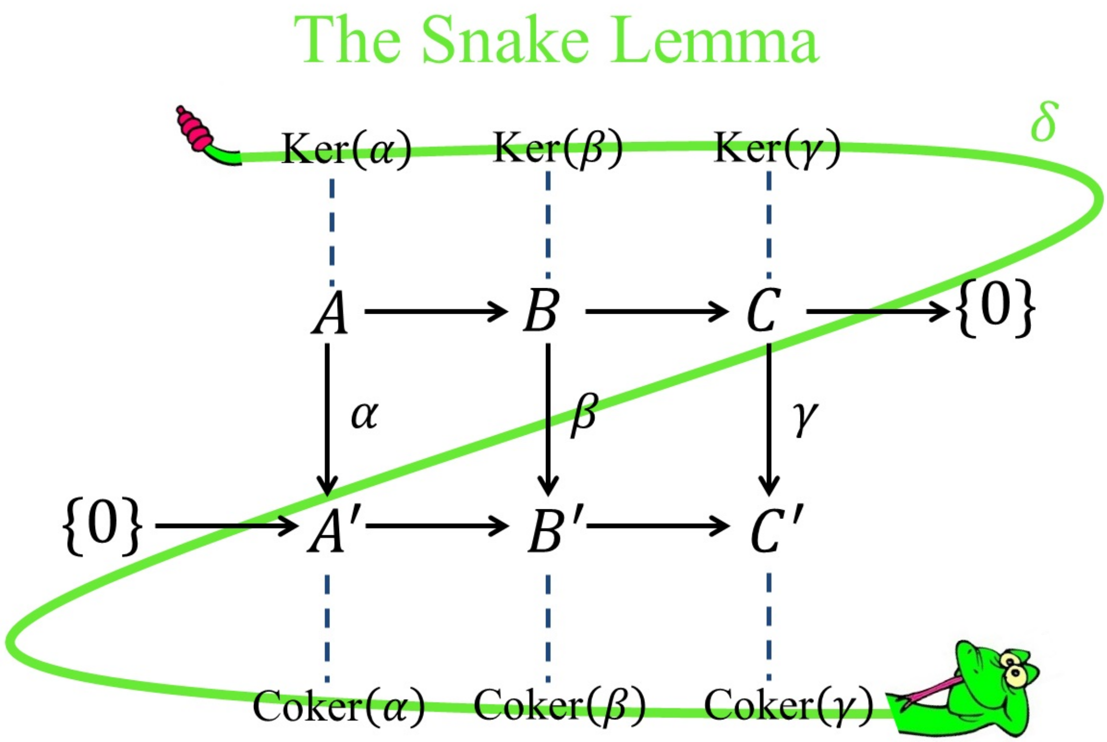
$$\begin{tikzcd} & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' & \end{tikzcd}$$

a commutative diagram of R-modules and R-module homomorphisms such that each row is an exact sequence. Then there is an exact sequence

$$\mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha) \to \mathrm{Coker}(\beta) \to \mathrm{Coker}(\gamma).$$

If in addition, $f_A : A \to B$ is a monomorphism then so is the homomorphism $k_\alpha : A' \to B'$, and if $g_{B'} : B' \to C'$ is an epimorphism then so is $b_\beta : \mathrm{Coker}(\beta) \to \mathrm{Coker}(\gamma)$. Under these added conditions, we can extend the exact sequence on the left to include "$0 \to$" and on the right to include "$\to 0$."

A proof of the Snake Lemma is given in Supplement. A Proof of The Snake Lemma. The name the "Snake Lemma" comes from how the exact sequence relates to the diagram given in the statement of the result. In cartoon form, we have:



Recall that $\mathrm{Coker}(\alpha) = A'/\mathrm{Im}(\alpha)$, $\mathrm{Coker}(\beta) = B'/\mathrm{Im}(\beta)$, and $\mathrm{Coker}(\gamma) = C'/\mathrm{Im}(\gamma)$.

Definition. Two short exact sequences are *isomorphic* if there is a commutative diagram of module homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

such that f, g, and h are R-module isomorphisms.

Theorem IV.1.18. Let R be a ring and $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ a short exact sequence of R-module homomorphisms. Then the following conditions are equivalent:

(i) There is an R-module homomorphism $h : A_2 \to B$ with $gh = 1_{A_2}$;

(ii) There is an R-module homomorphism $k : B \to A_1$ with $kf = 1_{A_1}$;

(iii) the given sequence is isomorphic (with identity maps on A_1 and A_2) to the direct sum short exact sequence $\{0\} \to A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to \{0\}$; in particular $B \cong A_1 \oplus A_2$.

Definition. A short exact sequence that satisfies the equivalent conditions of Theorem IV.1.18 is *split* or is a *split exact sequence.*

Revised: 12/30/2023