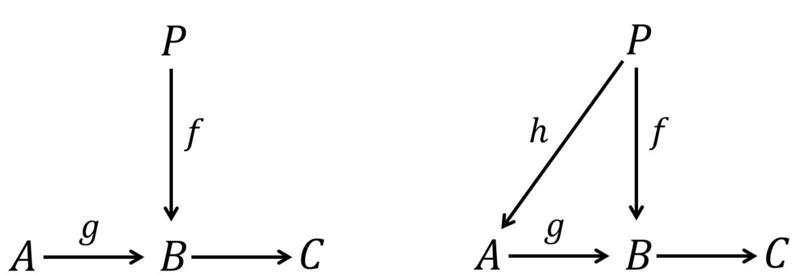
Section IV.3. Projective and Injective Modules

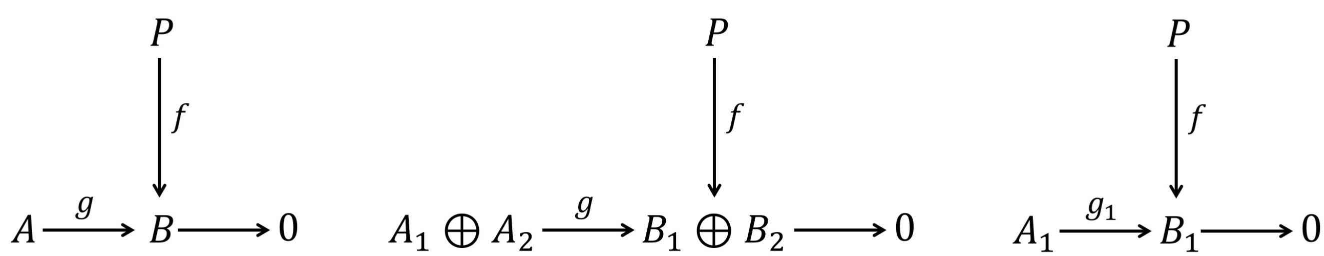
Note. In this section we define projective modules in terms of modules, homomorphisms, and exact pairs of module homomorphisms. Since this definition depends only on modules (the "objects" of interest) and homomorphisms (the "morphisms" on the objects), the idea of projective modules will be useful in the category setting. Just as we had dual statements in the category setting that resulted by reversing the "arrows" in a statement (see Section I.7. Categories: Products, Coproducts, and Free Objects), the dual of projectivity is injectivity and the idea of an injective module.

Note. Recall that for A, B, C modules over a ring R and $f : A \to B$, $g : B \to C$ R-module homomorphisms, that is $A \xrightarrow{f} B \xrightarrow{g} C$, the pair of homomorphisms is *exact* if $\text{Im}(f) = \text{Ker}(g)$.

Definition IV.3.1. A module P over a ring R is *projective* if given any diagram of R-module homomorphisms (below left) with bottom row $A \xrightarrow{g} B \to 0$ exact (that is, g is an epimorphism [onto]), there exists an R-module homomorphism $h : P \to A$ such that the diagram (below right) is commutative (that is, $gh = f$).



Note IV.3.A. Suppose R is a ring with identity and A, B are R-modules. Then by Exercise IV.1.17(a) there are submodules A_1 and B_1 of A and B, respectively, such that A_1 and B_1 are unitary, $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ with $RA_2 = RB_2 = 0$. Suppose for unitary R-module P we have R-module homomorphism $f : P \to B$. By Exercise IV.1.17(b), $f(R) \subset B_1$. Let g be an R-module homomorphism with $g : A \to B$. Then by Exercise IV.1.17(c), if g is an epimorphism then both $g|_{A_1} : A_1 \to B_1$ and $g|_{A_2} : A_2 \to B_2$ are epimorphisms. Consider the three diagrams:



Let $h : P \to A$ be the R-module homomorphism such that $gh = f$. Again by Exercise IV.1.17(b) we have $h(P) \subset A_1$. So the claim $gh = f$, or $g(h(P)) = f(P) \subset B_1$ is the claim that for all $p \in P$ we have $g(h(p)) = f(p)$ or, since $h(p) \in A_1$, $f(p) \in B_1$, and $g : A_1 \to B_1$, we have $g|_{A_1}(h(p)) = f(p)$. So the elements of A_2 and B_2 (the "nonunitary" parts of A and B, if you like) play no role in the claim that $gh = f$. So if we can chow that $h_1 : P \to A_1$ exists such that for epimorphism $g_1 : A_1 \to B_1$ (in the above diagram on the right) we have $g_1 g_1 = f$, then we can take $h = h_1$ to give the desired function f to establish that P is projective (and conversely, we can take $g_1 = g|_{A_1}$ if we are given the diagram on the left and/or center above); notice there is no claim of surjectivity except for g. That is, without loss of generality, we can show that unitary R-module P is projective by assuming R-modules A and B are unitary.

Note. Our first result gives us a family of examples of projective modules by showing that every free module over a ring with identity is projective.

Theorem IV.3.2. Every free module F over a ring with identity is projective.

Note IV.3.B. Theorem IV.3.2 holds if we drop the condition of R having an identity and require F to be a free module in the category of all left R-modules, as considered in Note IV.2.D and Exercise IV.2.2. The proof is the same as given for Theorem IV.3.2, but with Exercise IV.2.2 replacing the use of Theorem IV.2.1 (and dropping "unitary," of course). This is the sense in which the following corollary is stated. We can insert the condition that R has an identity in the corollary and use the definition of "free R-module" based on Theorem IV.2.1.

Corollary IV.3.3. Every module A over a ring R in the homomorphic image of a projective module.

Note IV.3.C. The next result classifies projective R-modules in terms of short exact and split exact sequences, and in terms of direct sums. Recall that a *short exact sequence* is one of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ where f is a monomorphism, g is an epimorphism, and $\text{Im}(f) = \text{Ker}(g)$ (see Note IV.1.I). A short exact sequence is split exact if there is R-module homomorphism $h : C \to B$ with $gh = 1_C$ (or one of the other equivalent conditions given in Theorem IV.1.18). Part (iii) of the next result refers to a free module F. This may be a free module either in the sense of

Theorem IV.2.1 (though this case requires R to have an identity and module P of the next result to be unitary) or in the sense of Note IV.2.D/Exercise IV.2.2.

Theorem IV.3.4. Let R be a ring. The following conditions on R-module P are equivalent.

(i) P is projective;

(ii) every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split exact (hence $B \cong A \oplus P$);

(iii) there is a free module F and an R-module K such that $F \cong K \oplus P$.

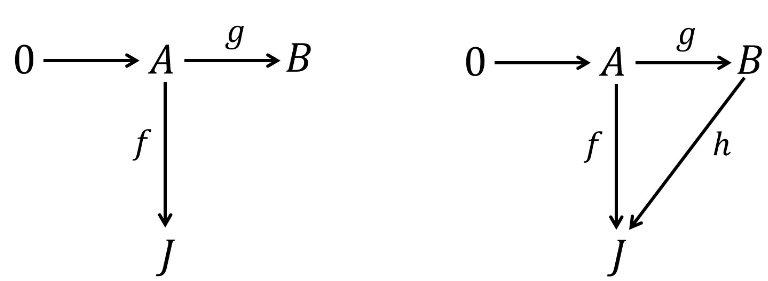
Note. $\mathbb{Z}_6$ is a free $\mathbb{Z}_6$-module (with basis $\{\bar{1}\}$, say). By Exercise IV.1.1, $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are $\mathbb{Z}_6$-modules. Since $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$ then by Theorem IV.3.4 (iii) $\Rightarrow$ (i), we have that $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are projective as $\mathbb{Z}_6$-modules. We saw in Note IV.2.H that $\mathbb{Z}_3 \cong \{\bar{0}, \bar{2}, \bar{4}\}$ is not a free $\mathbb{Z}_6$-module (since there can be no linearly independent subset of $\{\bar{0}, \bar{2}, \bar{4}\}$; similarly $\mathbb{Z}_2 \cong \{\bar{0}, \bar{3}\}$ is not a free $\mathbb{Z}_6$-module). So we see by example that the R-module K and P for which $F = K \oplus P$ (in (iii) of Theorem IV.3.4) need not themselves for free R-modules.

Note. We have now completed the part of the section necessary to cover Section IV.6. Modules over a Principal Ideal Domain. So if the priority is to cover Chapter VII, "Linear Algebra," in a graduate-level Linear Algebra class, then the remainder of this section can be skipped.

Proposition IV.3.5. Let R be a ring. A direct sum of R-modules $\sum_{i \in I} P_i$ is projective if and only if each P_i is projective.

Note IV.3.D. Recall that the "dual" of a statement in category theory results by "reversing arrows" in diagrams (see Section I.7. Categories: Products, Coproducts, and Free Objects and particularly the definition of "coproduct"). In this spirit, we "might say" the dual of an epimorphism mapping $A \to B$ is a monomorphism mapping $B \to A$. This is imprecise (and incorrect if we think in terms of inverse functions, but that's not the intent) but is motivated by the fact that $B \to A$ is an epimorphism if and only if $B \to A \to 0$ is exact, and $A \to B$ is a monomorphism if and only if $0 \to A \to B$ is exact. We mimic the definition of projective module, but with an attempt at duality.

Definition IV.3.6. A module J over a ring R is *injective* if given any diagram of R-module homomorphisms (below left) with top row exact (i.e., g is a monomorphism), there exists an R-module homomorphism $h : B \to J$ such that the diagram (below right) is commutative (that is, $hg = f$).



Note IV.3.E. The observations of Note IV.3.A concerning projective R-modules hold for injective modules as well. That is, without loss of generality, we can show that unitary R-module J is injective by assuming R-modules A and B are unitary.

Note. Recall that in categories the dual concept of a direct sum (also called a "coproduct") is a direct product. Some (but not all) of the results on projective modules have dual results on injective modules. The dual of Proposition IV.3.5 is the following, the proof of which we leave as Exercise IV.3.A.

Proposition IV.3.7. A direct product of R-modules $\prod_{i \in I} J_i$ is injective if and only if J_i is injective for every $i \in I$.

Note IV.3.F. It is to be shown in Exercise IV.3.13 that there is no dualized version of the concept of a free module (such a module, if it exists, would be called "co-free"). Since Theorem IV.3.2 and part (iii) of Theorem IV.3.4 refer to free modules in the projective module setting, they do not have duals in the injective module setting. However, Corollary IV.3.2 does have a dual version in which it is claimed that every module can be embedded in an injective module. This is proved in Proposition IV.3.12, after presenting the proofs of four preliminary lemmas. The dual version of parts (i) and (ii) of Theorem IV.3.4 is also given below in Proposition IV.3.13. This proof is not needed in the remainder of the course, so we just state the lemmas and give a little commentary.

Lemma IV.3.8. Let R be a ring with identity. A unitary R-module J is injective if and only if for every left ideal L of R, any R-module homomorphism $L \to J$ may be extended to an R-module homomorphism $R \to J$.

Definition. An abelian group D is *divisible* if given any $y \in D$ and $0 \neq n \in \mathbb{Z}$, there exists $x \in D$ such that $nx = y$.

Note. It is to be shown in Exercise IV.3.4 that $\mathbb{Q}$ is divisible but $\mathbb{Z}$ is not. In Exercise IV.3.7 it is to be shown that the homomorphic image of a divisible group is divisible (part (a) and the direct sum of abelian groups is divisible if and only if each summand is divisible (parts (b) and (c)).

Lemma IV.3.9. An abelian group D is divisible if and only if D is an injective (unitary) $\mathbb{Z}$-module.

Note. Divisible abelian groups (and hence injective unitary $\mathbb{Z}$-modules, by Lemma IV.3.9) are classified in Exercise IV.3.11: Every divisible abelian group is a direct sum of copies of $\mathbb{Q}$ and copies of $Z(p^\infty)$ for various primes p (see Exercise I.1.10 where $Z(p^\infty)$ is defined as the subgroup of $\mathbb{Q}/\mathbb{Z}$ given by

$$Z(p^\infty) = \{\overline{a/b} \in \mathbb{Q}/\mathbb{Z} \mid a, b \in \mathbb{Z}, b = p^i \text{ for some } i \geq 0\}.$$

Lemma IV.3.10. Every abelian group A may be embedded in a divisible abelian group.

Lemma IV.3.11. If J is a divisible abelian group and R is a ring with identity, then $\mathrm{Hom}_{\mathbb{Z}}(R, J)$ is an injective left R-module.

Proposition IV.3.12. Every unitary module A over a ring R with identity may be embedded in an injective R-module.

Proposition IV.3.13. Let R be a ring with identity. The following conditions on a unitary R-module J are equivalent.

(i) J is injective;

(ii) every short exact sequence $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split exact (hence $B \cong J \oplus C$);

(iii) J is a direct summand of any module B of which it is a submodule.

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