Section IV.6. Modules Over a Principal Ideal Domain

Note. This section is largely only necessary for Sections VII.2, "Rank and Equivalence," and VII.4, "Decompositions of a Single Linear Transformation and Similarity" in Chapter VII on Linear Algebra (see my online notes, in preparation, on a graduate level Linear Algebra class). We'll concentrate on the structure of all finitely generated modules over a principal ideal domain. In the process, we carry over most of the results on finitely generated abelian groups from Section II.1. Free Abelian Groups and Section II.2. Finitely Generated Abelian Groups to our setting. In Theorem IV.6.12 and Corollary IV.6.13 we give a type of decomposition of finitely generated modules over a PID which allow us to give invariants shared by all isomorphic modules. **By convention, in this section all modules are assumed to be unitary modules!!!**

Note. A free (unitary) module over a principal ideal domain (with identity) R has the invariant dimension property by Corollary IV.2.12 (the "in particular" part; since R is a PID then it is an integral domain and so is commutative). So the rank of a free R-module (given an Definition IV.2.8; recall that by convention we are using the term "dimension" for a vector space and "rank" for other moudles) is well-defined. By Proposition IV.2.9, two free R-modules are isomorphic if and only if they have the same rank (recall that the rank involves a cardinal number, so we certainly are not restricted to finite rank free modules). The next result is a generalization from the setting of finite rank free abelain groups (see Theorem II.1.6) to free modules over a PID.

Theorem IV.6.1. Let F be a free module over a principal ideal domain R and G a submodule of F. Then G is a free R-module and $\text{rank}(G) \leq \text{rank}(F)$.

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