Section IV.7. Algebras

Note. In this section, we define a K-algebra over a commutative ring K, and we define a division algebra. We give examples of such structures, define a subalgebra, algebra ideal, and K-algebra homomorphisms. These results require little background (beyond a knowledge of rings and an introduction to modules). We present two theorems, but they require a knowledge of tensor products, as given in Section IV.5. We'll see algebras again in Section IX.5 where we introduce algebra modules and algebraic algebras. In Section IX.6, we introduce division algebras and prove in Frobenius' Theorem (one of the last results in the book concerning algebra, as opposed to Chapter XI which covers category theory) that the only algebraic division algebras over the field $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ (the real quaternions).

Definition IV.7.1. Let K be a commutative ring with identity. A *K-algebra* (or *algebra over K*) A is a ring A such that:

(i) $(A, +)$ is a unitary (left) K-module;

(ii) $k(ab) = (ka)b = a(kb)$ for all $k \in K$ and $a, b \in A$.

A K-algebra A which, as a ring, is a division ring, is called a *division algebra.*

Note. Since $(A, +)$ is a (left) K-module, then the action of commutative ring K on ring A is defined (see Definition IV.1.1) and satisfies distribution and associativty. Notice that a K-algebra A is a ring but A may not be commutative even though ring K is.

Note IV.7.A. Since K is commutative, every left K-module A is also a right K module with the definition $ka = ak$ for all $a \in A$ and $k \in K$ (commutivity is needed to satisfy property (iii) of the definition of module, Definition IV.1.1). So every left K-algebra is also a right K-algebra (again, with the definition $ka = ak$).

Note. Hungerford states (page 227) that: "The classical theory of algebras deals with algebras over a field K." Such an algebra is actually a vector space (notice that the algebra has the added structure of vector multiplication since the "vectors" are from ring A). If an algebra over a field is finite dimensional as a vector space over K then it is a *finite dimensional algebra* over K.

Example IV.7.A. Every ring R is an additive abelian group and hence is a $\mathbb{Z}$-module. For $n \in \mathbb{Z}$ and $a, b \in R$ we have

$$n(ab) = \underbrace{ab + ab + \cdots + ab}_{n \text{ times}} = \begin{cases} \underbrace{(a + a + \cdots + a)}_{n \text{ times}} b = (na)b \\ a \underbrace{(b + b + \cdots + b)}_{n \text{ times}} = a(nb) \end{cases}$$

(with the obvious meaning in terms of additive inverses when $n < 0$), so (ii) of Definition IV.7.1 holds and hence every ring R is a $\mathbb{Z}$-module and a $\mathbb{Z}$-algebra.

Examples. If K is a commutative rings with identity, then the polynomial ring $K[x_1, x_2, \ldots, x_n]$ and the power series ring $K[[x]]$ are K-algebras where the K-module action is defined in the usual way of multiplication of polynomials/series by "constants."

Example. Let A be a ring with identity and K a subring of the center of A such that $a_A \in K$ (so that $(A, +)$ is a unitary K-module). Then A is a K-algebra, with the K-module action given by multiplication in A. In particular, every commutative ring K with identity is a K-algebra.

Example. The complex numbers $\mathbb{C}$ are a $\mathbb{R}$-algebra; it is a finite dimensional algebra over $\mathbb{R}$ of dimension 2. The quaternions $\mathbb{H}$ are a (noncommutative) $\mathbb{R}$-algebra and a finite dimensional algebra over $\mathbb{R}$ of dimension 4.

Example. Let G be a multiplicative group and K a commutative ring with identity. We defined the group ring $K(G)$ in Example III.1.A. The elements of $K(G)$ are $|G|$-tuples of elements of K where all but finitely many entries are 0, so the elements of $K(G)$ form the set denoted $\sum_{g\in G} R$. For $x = \{k_g\}_{g\in G} \in K(G)$ we denote x as the formal sum

$$x = k_{g_1}g_1 + k_{g_2}g_2 + \cdots + k_{g_n}g_n = \sum_{i=1}^{n} r_{g_i}g_i$$

where the nonzero k_g in x are $k_{g_1}, k_{g_2}, \ldots, k_{g_n}$. The group ring $K(G)$is actually a K-algebra where the K-module action is given by $k\left(\sum r_i g_g\right) = \sum (kr_i)g_i$ where $k, r_i \in K$ and $g \in G$. $K(G)$ is the *group algebra* of G over K.

Example. If K is a commutative ring with identity then the ring $\mathrm{Mat}_n(K)$ of all $n \times n$ matrices over K is a K algebra with the K-module action given as the usual scalar multiplication of a matrix so that for $k \in K$ and $A, B \in \mathrm{Mat}_n(K)$ we have $k(AB) = (kA)B = A(kB)$. In fact, if A is a K-algebra, then so is $\mathrm{Mat}_n(A)$.

Note. The next theorem gives an equivalent condition for a unitary left K-module to be a K-algebra. So it yields an alternative definition of a K-algebra and it does so in terms of tensor products. The motivation for the next theorem is the fact that for any ring R, there is a unique map (by Theorem IV.5.2) mapping $R \otimes_{\mathbb{Z}} R \to R$ (defined on a generator $r \otimes s$ by $r \otimes s \mapsto rs$) is a homomorphism of additive abelian groups. Since rings are $\mathbb{Z}$-algebras (by Example IV.7.A), this observation is a special case of the next theorem.

Theorem IV.7.2. Let K be a commutative ring with identity and A a unitary let K-module. Then A is a K-algebra (note that is A is a left K-algebra then A is also a right K-algebra by Note IV.7.A) if and only if there exists a K-module homomorphism $\pi : A \otimes_K A \to A$ such that the following diagram is commutative:

$$\begin{array}{ccc} A \otimes_K A \otimes_K A & \xrightarrow{\pi \otimes 1_A} & A \otimes_K A \\ \downarrow {\scriptstyle 1_A \otimes \pi} & & \downarrow {\scriptstyle \pi} \\ A \otimes_K A & \xrightarrow{\pi} & A \end{array}$$

In this case, the K-algebra A has an identity if and only if there is a K-module homomorphism $I : K \to A$ such that the following diagram is commutative:

$$\begin{array}{ccccc} K \otimes_K A & \xrightarrow{\zeta} & A & \xleftarrow{\theta} & A \otimes_K K \\ \downarrow {\scriptstyle I \otimes 1_A} & & \downarrow {\scriptstyle 1_A} & & \downarrow {\scriptstyle 1_A \otimes I} \\ A \otimes_K A & \xrightarrow{\pi} & A & \xleftarrow{\pi} & A \otimes_K A \end{array}$$

where ζ and θ are the isomorphisms of Theorem IV.5.7.

Definition. The homomorphism $\pi : A \otimes_K A \to A$ of Theorem IV.7.2 is the *product map* of the K-algebra A. The homomorphism I, where $I \otimes 1_A : K \otimes_K A \to A \otimes_K A$ and $1_A \otimes I : A \otimes_K K \to A \otimes_K A$, is the *unit map*.

Note. We now define subalgebra, ideal, homomorphism, and isomorphism for K-algebras. We make use of the corresponding definitions of modules.

Definition IV.7.3. Let K be a commutative ring with idenity and A and B be K-algebras.

(i) A *subalgebra* of A is a subring of A that is also a K-submodule of A.

(ii) A (left, right, two-sided) *algebra ideal* of A is a (left, right, two-sided, respectively) ideal of the ring A that is also a K-submodule of A.

(iii) A *homomorphism* (respectively, *isomorphism*) *of K-algebras* $f : A \to B$ is a ring homomorphism (respectively, isomorphism) that is also a K-module homomorphism (respective, isomorphism).

Note. if A is a K-algebra where A does not have an identity, then an ideal of ring A may not be an algebra ideal of ring A, as is to be shown by example in Exercise IV.7.4. However, if A has an identity then we have the following.

Theorem IV.7.A. If A is a K-algebra and ring A has an identity, then a (left, right, two-sided) ideal of ring A is also a (left, right, two-sided, respectively) algebra ideal of K-algebra A.

Note. Just as we have defined quotient groups using normal subgroups, and quotient rings using ideals, we can define quotient algebras of a K-algebra using algebra ideals (in the obvious way, using cosets).

Note. We can define direct products and direct sums of families of K-algebras to produce new K-algebras (as we did for rings in Section III.2, "Ideals"). We can also use tensor products to construct new algebras, as will be shown in the next theorem.

Note. It is to be shown in Exercise IV.7.2 that if A and B are K-modules then there is a K-module isomorphism $\alpha : A \otimes_K B \to B \otimes_K A$ such that $\alpha(a \otimes b) = b \otimes a$ for all $a \in A$ and $b \in B$. This useful in the proof of the following (which we leave as Exercise IV.7.A).

Theorem IV.7.4. Let A and B be algebras (with identity) over a commutative ring K with identity. Let π be the composition

$$(A \otimes_K B) \otimes_K (A \otimes_K B) \xrightarrow{1_A \otimes \alpha \otimes a_B} (A \otimes_K A) \otimes_K (B \otimes_K B) \xrightarrow{\pi_A \otimes \pi_B} A \otimes_K B,$$

where π_A and π_B are the product maps of A and B, respectively. Then $A \otimes_K B$ is a K-algebra (with identity) with product map π.

Note. We now define the tensor products of K-algebras. This plays a large role in the study of division rings in Section IX.6 (the last section of the book covering topics from modern algebra).

Definition. Let K be a commutative ring with idenity and let A and B be K-algebras. The K-algebra $A \otimes_K B$ of Theorem IV.7.4 is the *tensor product of K-algebras* A and B.

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