Chapter IX. The Structure of Rings

Section IX.1. Simple and Primitive Rings

Note. This section of the text starts with a "big picture" conversation. We'll try to address this after introducing some of the new ideas. The background necessary for this section includes Sections IV.1, "Modules, Homomorphisms, and Exact Sequences," Section IV.2, "Free Modules and Vector Spaces," and Section VIII.1, "Chain Conditions."

Note. Exercise IV.1.7 introduces an *endomorphism ring*: "If A and B are R-modules, then the set $\text{Hom}_R(A, B)$ of all R-module homomorphisms mapping $A \to B$ is an abelian group with $f + g$ given on $a \in A$ by $(f + g)(a) = f(a) + g(a) \in B$. The identity element is the zero map. $\text{Hom}_R(A, A)$ is a ring with identity, where multiplication is composition of functions. $\text{Hom}_R(A, A)$ is the *endomorphism ring* of A." In this section we consider endomorphism rings where the R-module A is a vector space.

Definition IX.1.1. A (left) module A over a ring R is *simple* (or *irreducible*) provided $RA \neq \{0\}$ and A has no proper submodules. A ring R is *simple* if $R^2 \neq \{0\}$ and R has no proper (two-sided) ideals.

Note. Trivially, every simple module is nonzero (i.e., not just $\{0\}$) and similarly every simple ring is nonzero. A unitary module A over a ring R with identity has $RA \neq \{0\}$ (unless $A = \{0\}$; recall $1_R a = a$ for all $a \in A$ by Definition IV.1.1(iv), "unitary R-module," and so A is simple if and only if A has no proper submodules (notice this statement holds for $A = \{0\}$ also).

Example. If R is a ring with identity and A is an R-module, then by Exercise IV.1.17(a), there are submodules B and C of A such that B is unitary, $RC = \{0\}$, and $A = B \oplus C$. So if A is simple then it must be tat $C = \{0\}$ and $A = B$. That is, every simple module over a ring with identity is unitary.

Lemma IX.1.A. Every simple module A is cyclic. In fact, $A = Ra$ for every nonzero $a \in A$.

Note. The converse of Lemma IX.1.A does no hold since $\mathbb{Z}$ module $\mathbb{Z}_6$ is cyclic (it is generated by $\overline{1}$), but is has proper submodules $\{\overline{0}, \overline{2}, \overline{4}\}$ and $\{\overline{0}, \overline{3}\}$.

Example. A division ring D (where $D \neq \{0\}$) has all nonzero elements as units and so every nonzero ideal must contain 1_D (since ideals are subrings). But then the ideal must equal D (since $d \in D$ and $1_D \in I$ implies $dt_D = d \in I$). so D has no proper ideal and division ring D must be simple. Similarly, D as a D-module is simple.

Example. Let D be a division ring and let $R = \text{Mat}_n(D)$ be the ring of all $n \times n$ matrices with zero columns except possible column k). It was shown in the proof of Corollary VIII.1.2 that I_k (denoted as Re_k in the proof) is a left ideal of R which is nonzero and has no proper submodules. That is, each I_k is a simple left R-module.

Example. Let D be a division ring and let $R = \text{Mat}_n(D)$ where $n > 1$. By the previous example, I_k is a proper left ideal of R and so $\text{Mat}_n(D)$ is not a simple left *module* over itself. But as a *ring*, $\text{Mat}_n(D)$ (here we take $n \geq 1$) has no proper ideals and so as a ring $\text{Mat}_n(D)$ is simple. So the simplicity of a ring R may differ from the simplicity of ring R treated as an R-module.

Definition. A left ideal of a ring R is a *minimal left ideal* if $I \neq \{0\}$ and for every left ideal J of R such that $\{0\} \subset J \subset I$, either $J = \{0\}$ or $J = I$.

Note. By the definition of "simple ring" (Definition IX.1.1), a left ideal I of ring R such that $RI \neq \{0\}$ is a simple left R-module if and only if I is a minimal left ideal.

Lemma IX.1.B. Let $A = Ra$ be a cyclic R-module. Define $\theta : R \to A$ as $\theta(r) = ra$. Then $R/\text{Ker}(\theta)$ (and hence A) has no proper submodules if and only if $\text{Ker}(\theta)$ is a maximal left ideal of R.

Note IX.1.A. Let A be a simple R-module. By Lemma IX.1.A, A is cyclic and $A = Ra$ for any nonzero $a \in A$. By Lemma IX.1.B (see the proof), since A is simple, $A \cong R/I$ where I is some maximal left ideal (in fact, I is the kernel of some homomorphism). So every simple R-module is isomorphic to R/I for some maximal left ideal I. We now introduce a condition that will allow us to show that the converse of this holds for a certain class of maximal left ideals.

Definition IX.1.2. A left ideal I in a ring R is *regular* (or *modular*) if there exists $e \in R$ such that $r - re \in I$ for every $r \in R$. Similarly, a right ideal J is *regular* if there exists $e \in R$ such that $r - er \in J$ for every $r \in R$.

Note. If R is a ring with identity 1_R, then every ideal is regular since we can take $e = 1_R$ and then $r - re = r - er = 0$, and 0 is in every ideal. Now we relate simple R-modules to quotients R/I where I is a maximal *regular* ideal.

Theorem IX.1.3. A left module A over ring R is simple if and only if A is isomorphic to R/I for some regular maximal left ideal I. This holds also if we replace "left" with "right."

Note. We need one more result before defining a primitive ring. In the Wedderburn-Artin Theorem (Theorem IX.1.14) we'll see these ideas united.

Theorem IX.1.4. Let B be a subset of a left module A over a ring R. Then $\mathcal{A}(B) = \{r \in R \mid rb = 0 \text{ for all } b \in B\}$ is a left ideal of R. If B is a submodule of A, then $\mathcal{A}(B)$ is an (two sided) ideal.

Definition. Let B be a subset of a left module A over a ring R. Left ideal $\mathcal{A}(B) = \{r \in R \mid rb = 0 \text{ for all } b \in B\}$ is the left *annihilator* of B. The right annihilator of a right module is similarly defined.

Definition IX.1.5. A left module A is *faithful* if its left annihilator $\mathcal{A}(A)$ is $\{0\}$. A ring R is *left primitive* if there exists a simple faithful left R-module. A right primitive ring is similarly defined.

Note. There exist right primitive rings that are not left primitive; see G. Bergman, "A Ring Primitive on the Right but Not on the Left," *Proceedings of the American Mathematical Society* **15**, 473–475 (1964); a correction is given on page 1000.

Note. From now on we use the term "primitive" to mean "left primitive." All results proved for left primitive rings are true also for right primitive rings.

Example. [This example requires two results from Section IV.2, "Free Modules and Vector Spaces"; namely, Theorems IV.2.1 and IV.2.4.] Let V be a (possibly infinite dimensional vector space over a division ring D (see Definition IV.1.1) and let R be the endomorphism ring $\mathrm{Hom}_D(V, V)$ of V (see Exercise IV.1.7). By Exercise IV.1.7(c), V is a left R-module (or "$\mathrm{Hom}_D(V, V)$-module") with $\theta v = \theta(v)$

for all $v \in V$ and $\theta \in R$. If u is a nonzero vector in V, then there is a basis of V that contains u by Theorem IV.2.4. For each $v \in V$ we define $\theta_v \in R = \mathrm{Hom}_D(V,V)$ by defining $\theta_v(u) = v$ and $\theta_v(w) = 0$ for all basis elements x except v; this determines θ_v on all of V and θ_v is in fact a homomorphism (by Theorem IV.2.4, V is a free D-module; by Theorem IV.2.1(iv), θ_v is a homomorphism [see the proof of (i) implies (iv)]). Now

$$Ru = \mathrm{Hom}_D(V,V)\,u = \{\theta(u) \mid \theta \in \mathrm{Hom}_D(V,V)\} \supset \{\theta_v(u) \mid v \in V\} = V,$$

or $Ru = V$ and this holds for any given nonzero $u \in V$. Whence V has no proper R-submodules (since $Ru = V$ for all nonzero $u \in V$, the only possible submodule is $R0 = \{0\}$). Since $R = \mathrm{Hom}_D(V,V)$ has an identity (namely, the identity homomorphism), $RV \neq \{0\}$. So V is a simple (left) R-module (by Definition IX.1.1). If θ is in the annihilator of V, $\theta \in \mathcal{A}(V)$; that is, $\theta V = \{0\}$ for some $\theta \in R = \mathrm{Hom}_D(V,V)$ then "clearly" $\theta = 0$. So V is a faithful (left) R-module (by Definition IX.1.5). Therefore $R = \mathrm{Hom}_D(V,V)$ is primitive (also by Definition IX.1.5). We now argue that the simplicity of $R = \mathrm{Hom}_D(V,V)$ is determined by the dimension of V (finite versus infinite dimensional).

If V is of finite dimension n, then $R = \mathrm{Hom}_D(V,V)$ is isomorphic to a ring of all $n \times n$ matrices with entries from D by Theorem VII.1.4 (notice that any vector space V with a basis is a free D-module by the definition of free D-module; see Theorem IV.2.1) and by Exercise III.2.9(a), R has no proper ideals (and so no proper D-submodules). Now $R = \mathrm{Hom}_D(V,V)$ contains the identity matrix, so $RV \neq \{0\}$. So by Definition IX.1.1, ring $R = \mathrm{Hom}_D(V,V)$ is simple. If V is infinite dimensional, then it is to be shown that ring $R = \mathrm{Hom}_D(V,V)$ is not simple.

Note. In the previous example, the simplicity of ring $R = \text{Hom}_D(V, V)$ being related to the dimension of vector space V over division ring D (finite versus infinite dimensional) fore shadows the main result of this section, the Wedderburn-Artin Theorem (Theorem IX.1.14).

Note. In the previous example we saw that the ring $R = \text{Hom}_D(V, V)$, where V is a vector space, is primitive. We now give two results that let us find other examples of primitive rings.

Proposition IX.1.6. A simple ring R with identity is primitive.

Proposition IX.1.7. A commutative ring R is primitive if and only if R is a field.

Note. The classification of noncommutative primitive rings is more complicated. We accomplish this in the Jacobson Density Theorem (Theorem IX.1.12). We start with a definition.

Definition IX.1.8. Let V be a (left) vector space over a division ring D. A subring R of the endomorphism ring $\text{Hom}_D(V, V)$ (see Exercise IV.1.7) is a *dense ring of endomorphisms* of V (or a *dense subring* of $\text{Hom}_D(V, V)$) if for every $n \in \mathbb{N}$, every linearly independent subset $\{u_1, u_2, \ldots, u_n\}$ of V and every arbitrary subset $\{v_1, v_2, \ldots, v_n\}$ of V, there exists $\theta \in R$ such that $\theta(u_i) = v_i$ for $i = 1, 2, \ldots, n$.

Lemma IX.1.C/Example. For V a vector space over a division ring D, the endomorphism ring $\mathrm{Hom}_D(V, V)$ is a dense subring of itself.

Note. If V is finite dimensional then, in fact, the only dense subring of $\mathrm{Hom}_D(V, V)$ is itself, as the next theorem shows.

Theorem IX.1.9. Let R be a dense ring of endomorphisms of a vector space V over a division ring D. Then R is left (respectively, right) Artinian if and only if $\dim_D(V)$ is finite, in which case $R = \mathrm{Hom}_D(V, V)$.

Note. We prove that an arbitrary primitive ring is isomorphic to a dense ring of endomorphisms of a suitable vector space in the Jacobson Density Theorem (Theorem IX.1.12). We first need two lemmas.

Lemma IX.1.10. **(Schur)** Let A be a simple module over a ring R and let B be any R-module.

(i) Every nonzero R-module homomorphism $f : A \to B$ is a monomorphism (one to one);

(ii) every nonzero R-module homomorphism $f : B \to A$ is an epimorphism (onto);

(iii) the endomorphism ring $D = \mathrm{Hom}_R(A, A)$ is a division ring.

Note IX.1.B. Let A be a simple R-module. By Exercise IV.1.7(c), A is a left $\mathrm{Hom}_R(A,A)$-module with fa defined as $fa = f(a)$ for $a \in A$ and $f \in \mathrm{Hom}_R(A,A)$. Since A is simple then by Lemma IX.1.10(iii), $\mathrm{Hom}_R(A,A)$ is a division ring. So, by Definition IV.1.1, "vector space," A is a vector space over $\mathrm{Hom}_R(A,A)$.

Lemma IX.1.11. Let A be a simple module over a ring R. Consider A as a vector space over the division ring $D = \mathrm{Hom}_R(A,A)$. If V is a finite dimensional D-subspace of the D-vector space A and $a \in A \setminus V$, then there exists $r \in R$ such that $ra \neq 0$ and $rV = 0$.

Note. We now have the equipment to classify primitive rings in terms of dense rings of endomorphisms of a vector space (notice that the vector space depends on a faithful simple R-module A which exists according the the definition of "primitive"). y Note IX.1.B (and Lemma IX.1.10(iii))m $D = \mathrm{Hom}_R(A,A)$ is a division ring and A is a vector space over D. So the only hypothesis of the Jacobson Density Theorem is that R is a primitive ring.

Theorem IX.1.12. Jacobson Density Theorem.
Let R be a primitive ring and A a faithful simple R-module. consider A as a vector space over the division ring $\mathrm{Hom}_R(A,A) = D$. Then R is isomorphic to a dense ring of endomorphisms of the D-vector space A.

Note. The only place the faithfulness of A is used in the proof of Jacobson's density Theorem is to show that homomorphism $\alpha$is a monomorphism (one to one). Weakening the hypothesis from ring R being primitive to just ring R having a simple R-module A, we can conclude the following.

Corollary IX.1.A. Let R be a ring such that A is a simple R-module. Consider A as a vector space over the division ring $\mathrm{Hom}_R(A, A) = D$. Then R has a homomorphic image that is a dense ring of endomorphisms of the D-vector space A.

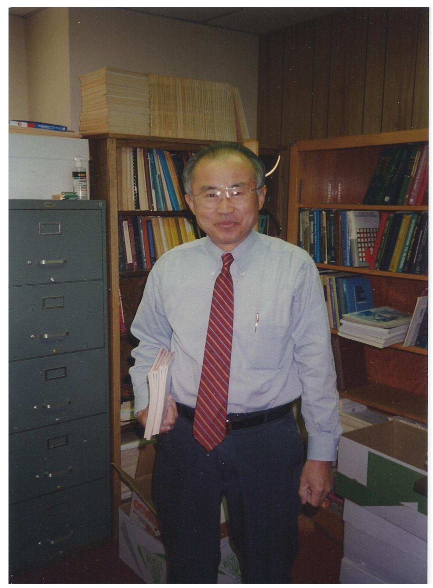
Note. A converse of the Jacobson Density Theorem (which weakens the condition of "dense ring of endomorphisms") to give in Exercise IX.1.4(a), as follows.

Theorem IX.1.A. Let V be a vector space over a division ring D. If R is a subring of $\mathrm{Hom}_D(V, V)$ such that for any $u, v \in V$ there exists $\theta \in R$ such that $\theta(u) = v$ (this is called 1-*fold transitive*, or just *transitive*), then R is primitive.

Corollary IX.1.13. If R is a primitive ring, then for some division ring D either R is isomorphic to the endomorphism ring of a finite dimensional vector space over D or for every $m \in \mathbb{N}$ there is subring R_m of R and an epimorphism of rings mapping $R_m \to \mathrm{Hom}_D(V_m, V_m)$ where V_m is an n-dimensional vector space over D.

Note. The Jacobson Density Theorem is named for Nathan Jacobson (October 5, 1910 to December 5, 1999). He was born in Warsaw, Poland and emigrated to the U.S. in 1918. He attended the University of Alabama as an undergraduate and got his Ph.D. from Princeton University in 1934. His Ph.D. advisor was Joseph Wedderburn. He worked at a number of U.S. universities, including University of Chicago, UNC-Chapel Hill, and Yale University. His work on the Density Theorem was published in: N. Jacobson, "Structure Theory of Simple Rings Without Finiteness Assumptions," *Transactions of the American Mathematical Society* **57**(2), 228–245 (1945). He wrote the (misleadingly titled) texts *Basic Algebra I* (NY: Freeman, 1974) and *Basic Algebra II* (NY: Freeman, 1980); Dr. Debra Knisley of ETSU used *Basic Algebra I* when teaching Modern Algebra 1 (MATH 5410) in the 1990s. A former ETSU Department of Mathematics member from the 1960s to the 1990s, Dr. Tae-Il Suh (June 1 1928 to July 27, 2009), was a student of Jacobson's (he was a 1961 graduate of Yale University).

Nathan Jacobson



Tae-Il Suh

Theorem IX.1.14. The Wedderburn-Artin Theorem for Simple Artinian Rings.

The following conditions on a left Artinian ring R are equivalent:

(i) R is simple;

(ii) R is primitive;

(iii) R is isomorphic to the endomorphism ring of a nonzero finite dimensional space V over a division ring D;

(iv) for some $b \in \mathbb{N}$, R is isomorphic to the ring of all $n \times n$ matrices over a division ring.

Note. We can think of the Wedderburn-Artin Theorem for Simple Artinian Rings (Theorem IX.1.14) as a classification of simple Artinian rings (or of primitive Artinian rings). We will see a second version of the Wedderburn-Artin Theorem in Theorem IX.2.2 when we give a classification of "semisimple" Artinian rings and a third version in Theorem IX.5.4 when we classify semisimple Artinian K-algebras.

Note. There is no claim of uniqueness of $\dim_D(V)$ in part (iii) nor of n in part (iv) in the Wedderburn-Artin Theorem for Simple Artinian Rings. However, uniqueness of both hold for given R, and the division rings of (iii) and (iv) are unique up to isomorphism, as we now show. We first need two lemmas.

Lemma IX.1.15. Let V be a finite dimensional vector space over a division ring D. If A and B are simple faithful modules over the endomorphism ring $R = \mathrm{Hom}_D(V, V)$, then A and B are isomorphic R-modules.

Lemma IX.1.16. Let V be a nonzero vector space over a division ring D and let R be the endomorphism ring $\mathrm{Hom}_D(V, V)$. If $g : V \to V$ is a homomorphism of additive groups such that $gr = rg$ for all $r \in R$, then there exists $d \in D$ such that $g(v) = dv$ for all $v \in V$.

Proposition IX.1.17. Let V_1 and V_2 be vector spaces of finite dimension n over the division rings D_1 and D_2, respectively.

(i) If there is an isomorphism of rings $\mathrm{Hom}_{D_1}(V_1, V_2) \cong \mathrm{Hom}_{D_2}(V_2, V_2)$, then $\dim_{D_1}(V_1) = \dim_{D_2}(V_2)$ and D_1 is isomorphic to D_2.

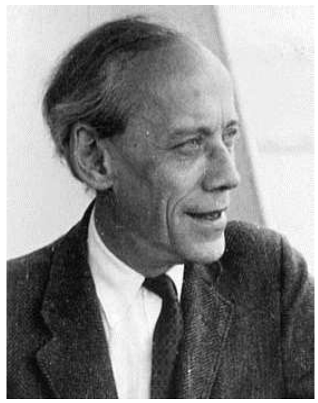
(ii) If there is an isomorphism of rings $\mathrm{Mat}_{n_1}(D_1) \cong \mathrm{Mat}_{n_2}(D_2)$, then $n_1 = n_2$ and D_1 is isomorphic to D_2.

Note. Joseph Wedderburn was born February 2, 1882 in Scotland. He was the 10th of 14 children. He studied at the University of Edinburgh, University of Leipzig, and University of Berlin, and University of Chicago. He spent most of his career with Princeton University where he was the Ph.D. advisor of Nathan Jacobson. Quoting from his Wikipedia page:

> "Wedderburn's best-known paper was his sole-authored 'On hypercomplex numbers,' published in the 1907 Proceedings of the London Mathematical Society, and for which he was awarded the D.Sc. the following year. This paper gives a complete classification of simple and semisimple algebras. He then showed that every semisimple algebra finite-dimensional can be constructed as a direct sum of simple algebras and that every simple algebra is isomorphic to a matrix algebra for some division ring. The ArtinWedderburn theorem generalises this result, with the ascending chain condition." [`https://en.wikipedia.org/wiki/Joseph_Wedderburn`, accessed 9/10/2018]

He served in the British army from 1914 to 1918. He retired in 1945 and lived mostly isolated until his death on October 9, 1948 at the age of 66.

Note. Emil Artin (March 3, 1898 to December 20, 1962) was born in Vienna, Austria. He studied at the University of Vienna, interrupted by service in the Austrian army in 1918. He spent 1921–22 at the University of Göttingen where he worked closely with Emmy Noether. From 1922 to 1937 he was at the University of Hamburg. His wife was half Jewish and he fled Germany in 1937 and took a job at the University of Notre Dame. The following year he moved to Indiana University. In 1946 he took a job at Princeton University and remained there until 1958 when he took a leave of absence. Over his career, he directed over 30 Ph.D. students, including Serge Lang and Hans Zassenhaus. One of his more influential books is the 68 page work, Galois Theory in the Notre Dame Mathematical Lectures, Number 2 (1942). Hungerford follows Artin's approach to Galois theory in Chapter V (also see my online notes for Section V.2, "The Fundamental Theorem (of Galois Theory)," `http://faculty.etsu.edu/gardnerr/5410/notes/V-2.pdf`. His idea of what we call "Artinian rings" is introduced in: E. Artin, C. Nesbitt, and R. Thrall, Rings with Minimum Condition, *University of Michigan Publications in Mathematics* **1**, Ann Arbor, Mich.: University of Michigan Press (1944). This work also includes his contribution to the Wedderburn-Artin Theorem. These biographical notes are based on the Wikipedia page at `https://en.wikipedia.org/wiki/Emil_Artin` (accessed 9/12/2018).



Note. The photographs in this section are from the MacTutor History of Mathematics Archive at `http://www-history.mcs.st-and.ac.uk/`, with the exception of the photograph of Dr. Tae-Il Suh which is from my personal collection.

Note. Finally, we return to Hungerford's "big picture" conversation mentioned at the beginnin of this section of notes. Hungerford states (page 415):

> "Matrix rings and endomorphism rings of vector spaces over division rings arise naturally in many different contexts. They are extremely useful mathematical concepts. Consequently it seems reasonable to take such rings, or at least rings that closely resemble them, as the basis of a structure theory and to attempt to describe arbitrary rings in terms of these basic rings."

The two fundamental properties we focused on in this section were simplicity and primitivity. We related these ideas to endomorphism rings of a vactor space, $\mathrm{Hom}_D(V, V)$. We have seen that a primitive ring is isomorphic to a dense subring of the endomorphism ring of a vector space over a division ring (The Jacobson Density Theorem, Theorem IX.1.12). A dense ring of endomorphisms of a vector space V (and hence a primitive ring) is Artinian if and only if $\dim_D(V)$ is finite (Theorem IX.1.9). A simple left Artinian ring is primitive (and a primitive left Artinian ring is simple) and is isomorphic to an endomorphism ring of a finite dimensional vector space (by the Wedderburn-Artin Theorem for Simple Artinian Rings, Theorem IX.1.14) and the dimension of the vector space is unique and the vector space is unique up to isomorphism (Theorem IX.1.17).

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