Section IX.2. The Jacobson Radical

Note. On page 425, Hungerford motivates "radicals" by introducing property P as an undesirable property. A P-radical $P(R)$ of a ring R is then an ideal that has property P and which contains all other ideal of R which have property P and for which $P/P(R)$ is zero. The desire is to find rings R for which the P-radical is zero, $P(R) = \{0\}$. Such a ring is "P-radical free" or P-semisimple.

Note. Joseph Wedderburn first introduced the idea of a radical; his idea only applied to (left) Artinian rings. His ideas have since been generalized to non-Artinian rings. In this section we define and explore the Jacobson radical; it will be used to define a "semisimple ring" (in Definition IX.2.9), and a version of the Wedderburn-Artin Theorem will be given for semisimple rings in the next section (in Theorem IX.3.3). The prime radical will be defined in Section IX.4 and used to define semiprime rings (classification for which are given in Propositions IX.4.2 and IX.4.4). We need two definitions before we define the Jacobson radical.

Definition IX.2.1. An ideal P of a ring R is said to be *left primitive* if the quotient ring R/P is a left primitive ring. Right primitive rings are similarly defined.

Note. R itself is an ideal of ring R. But $R/R \cong \{0\}$. Now $\{0\}$ has no simple left R-modules (see Definition IX.1.1 of "simple module") and so has no simple faithful left R-modules so that $\{0\}$ is not left primitive (see Definition IX.1.5). So $R/R \cong \{0\}$ is not left primitive and so ideal R of R is not left primitive.

Definition IX.2.2. An element a in a ring R is *left quasi-regular* if there exists $r \in R$ such that $r + a + ra = 0$. The element r is a *left quasi-inverse* of a. A (right, left, or two sided) ideal I of R is a *left quasi-regular* if every element of I is left quasi-regular. Similarly, $a \in R$ is *right quasi-regular* if there exists $r \in R$ such that $a + r + ar = 0$. Right quasi-inverses and right quasi-regular ideals are defined analogously.

Note IX.2.A. We adopt a set theoretic convention (Hungerford says this is a theorem of set theory) that: In a class $\mathcal{C}$ of those subsets of a ring R that satisfies a given property is empty, then $\cap_{i\in\mathcal{C}} I$ is defined to be R.

Theorem IX.2.3. If R is a ring, then there is an ideal $J(R)$ of R such that:

(i) $J(R)$ is the intersection of all the left annihilators of simple left R-modules;

(ii) $J(R)$ is the intersection of all the regular maximal left ideals of R;

(iii) $J(R)$ is the intersection of all the left primitive ideals of R;

(iv) $J(R)$ is a left quasi-regular left ideal which contains every left quasi-regular left ideal of R;

(v) Statements (i)–(iv) are also true if “left” is replaced by “right.”

Definition. The ideal $J(R)$ is the *Jacobson radical* of ring R.

Note. Hungerford mentions that historically part (iv) of Theorem IX.2.3 (the quasi-regularity part) was proved first but as the importance of modules in ring theory was realized, the first three parts of Theorem IX.2.3 appeared (the parts based on annihilators, regular maximal ideals, and left primitive ideals). We prove Theorem IX.2.3 below after presenting five lemmas.

Lemma IX.2.4. If I (where $I \neq R$) is a regular left ideal of a ring R, then I is contained in a maximal left ideal which is regular.

Lemma IX.2.5. Let R be a ring and let K be the intersection of all regular maximal left ideals of R. Then K is a left quasi-regular left ideal of R.

Lemma IX.2.6. Let R be a ring that has a simple left R-module. If I is a left quasi-regular left ideal R, then I is contained in the intersection of all the left annihilators of simple left R-modules.

Lemma IX.2.7. An ideal P of a ring R is left primitive if and only if P is the left annihilator of a simple left R-module.

Lemma IX.2.8. Let I be a left ideal of ring R. If I is left quasi-regular, then I is right quasi-regular.

Note. We now have the necessary equipment to give a proof of Theorem IX.2.3.

Note IX.2.B. If ring R has no left simple left R-modules (and so no left annihilators of simple left R-modules) must satisfy $J(R) = R$ by Theorem IX.2.3(i) and Note IX.1.A. If R has an identity then every ideal is regular (take $e = 1_R$ in the definition of "regular," Definition IX.1.2) and by Theorem III.2.18 maximal ideals always exist (and maximal ideals are proper subgroups of R by Definition III.2.17) so $J(R) \neq R$ by Theorem IX.2.3(ii).

Theorem IX.2.A. Let R be a commutative ring with identity which has a unique maximal ideal M (such a ring is a *local ring*; see Definition III.4.12). Then $J(R) = M$.

Example. As an application of Theorem IX.2.A, notice that the power series ring $F[[x]]$ over a field F is a local ring by Corollary III.5.10 and (also by Corollary III.5.10) the principal ideal (x) is maximal. So, by Theorem IX.2.A, $J(F[[x]]) = (x)$.

Example. For another application of Theorem IX.2.A, consider $\mathbb{Z}_{p^n}$ (for $n \geq 2$) where p is prime. Then $\mathbb{Z}_{p^n}$ is a local ring (it is commutative with identity and unique maximal ideal the principal ideal (p). So by Theorem IX.2.A, $J(\mathbb{Z}_{p^n}) = (p)$. Exercise IX.2.10 gives a result applicable to $J(\mathbb{Z}_m)$ for general $m \in \mathbb{N}$.

Definition IX.2.9. A ring R is (Jacobson) *semisimple* if its Jacobson radical $J(R) = \{0\}$. R is a *radical ring* if $J(R) = R$.

Note. Hungerford warns (on page 429): "Throughout this book 'radical' always means 'Jacobson radical' and 'semisimple' always means 'Jacobson semisimple.' " This is not a universal standard, since there are other types of radicals (other than the Jacobson radical), as Hungerford describes at the beginning of this section.

Example. Consider division ring D. If $I \neq \{0\}$ is an ideal of D then every nonzero element of I must have its multiplicative inverse in I (since I is a subring of D) and so $1_D \in I$. But then, $I = R$. So $I = \{0\}$ is the only maximal ideal of R (recall that maximal ideals are proper subrings of R; see Definition III.2.7). Since $1_D \in D$ then every ideal is a regular ideal of D (just take $e = 1_D$ in Definition IX.1.2), then the only regular maximal left ideal of D is $\{0\}$ and so by Theorem IX.2.3(ii), $J(R) = \{0\}$ and so D is semisimple. That is, every division ring is semisimple.

Example. Every maximal ideal in $\mathbb{Z}$ is of the form (p) where p is prime (by Theorem III.3.4). Since $\mathbb{Z}$ has an identity, then every ideal is regular (let e be the identity in Definition IX.1.2). So by Theorem IX.2.3(ii), $J(R) = \cap_{p \text{ prime}}(p) = \{0\}$ and hence $\mathbb{Z}$ is semisimple.

Example. We now show that if D is a division ring, then the polynomial ring $R = D[x_1, x_2, \ldots, x_m]$ is semisimple. If $f \in J(R)$ then by Theorem IX.2.3(iv) f is left quasi-regular and also right quasi-regular (by Theorem IX.2.3(v)). By Exercise IX.2.1, $q_R + f$ is a unit in R (and $1_R = 1_D$, so $1_D + f$ is a unit in R). By Theorem III.6.1, the units in $R = D[x_1, x_2, \ldots, x_m]$ are the nonzero elements of D (treated as constant polynomials in R) so $1_D + f \in D$ and hence $f \in D$ (since $-1_D \in D$). So $J(R) \subset D$ (that is, $J(R)$ consists only of constant polynomials) and $J(R)$ is an ideal of D (constant multiples of constant polynomials are constant polynomials). How as a division ring, D is a simple ring (see the Example in these class notes after Lemma IX.1.A) and so either $J(R) = \{0\}$ or $J(R) = R$. Now $a = -1_D$ is not left quasi-regular since for all $r \in R$ we have $r+a+ra = r+(-1_D)+r(-1_D) = -1_D \neq 0$ (see Definition IX.2.2, "left quasi-regular element") and so $-1_D \notin J(R)$ (using Theorem IX.2.3(iv)). Therefore $J(R) = \{)\}$ and R is semisimple.

Note. We now show interconnections between primitive, semisimple, and radical rings.

Theorem IX.2.10. Let R be a ring.

(i) If R is primitive, then R is semisimple.

(ii) If R is simple and semisimple, then R is primitive.

(iii) If R is simple, then R is either a primitive semisimple ring or a radical ring.

Theorem IX.2.B. Let D be a division ring. Then the ring of all $n \times n$ matrices over D, $\mathrm{Mat}_n(D)$, is semisimple.

Note. Wedderburn introduced the idea of a radical (in a left Artinian ring) as the maximal nilpotent ideal (not as the Jacobson radical, as we have used here). We now consider connections between Wedderburn's radical and the Jacobson radical.

Definition IX.2.11. An element a of a ring R is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$. A (left, right, two-sided) ideal I of R is *nil* if every element of I is nilpotent; ideal I is *nilpotent* if $I^n = \{0\}$ for some $n \in \mathbb{N}$ (here, I^n consists of all possible sums of products of n elements of I).

Note. A nilpotent ideal is nil since $I^n = \{0\}$ implies $a^n = 0$ for all $a \in I$. However, a nil ideal need not be a nilpotent ideal, as illustrated in Exercise IX.2.11.

Theorem IX.2.12. If R is a ring, then every nil right or left ideal is contained in the Jacobson radical $J(R)$.

Note. If a ring R is nil (that is, every element of R is nilpotent), then $J(R) = R$ and R is a radical ring.

Proposition IX.2.13. If R is a left (right) Artinian ring, then the radical $J(R)$ is a milpotent ideal. Consequently every nil left or right ideal of R is niplotent and $J(R)$ is the unique maximal nilpotent left (right) ideal of R.

Note. We now return to the beginning of this section and Hungerford's discussion if radicals in general. We start with a somewhat informal definition.

"Definition." Let P be a property of rings and an ideal (or ring) I is a *P-deal* (or *P-ring*) if I has property P. Assume that:

(i) the homomorphism image of a P-ring is a P-ring;

(ii) every ring R (or at least every ring in some specified class $\mathcal{C}$) contains a P-ideal $P(R)$ (called the *P-radical* of R) that contains all other P-ideals of R;

(iii) the P-radical of the quotient ring $R/P(R)$ is zero; and

(iv) the P-radical of ring $P(R)$ is $P(R)$.

A property P that satisfies (i)–(iv) is a *radical property.*

Note. Here, we take property P to be left quasi-regularity. By Theorem IX.2.3(iv), $J(R)$ is a left quasi-regular ideal which contains every left quasi-regular left ideal of R, so that part (ii) of "Definition" is satisfied. A ring homomorphism necessarily maps left quasi-regular elements onto left quasi-regular elements, so the homomorphic image of a P-ring (that is, a radical ring R for which $J(R) = R$) is a P-ring, and part (i) of "Definition" is satisfied. For property (iii) of "Definition," we need to show that $R/J(R) = \{0\}$ (i.e., R is semisimple; see definition IX.2.9), which we do in the following Theorem IX.2.14. For property (iv) of "Definition," we need to show that $J(J(R)) = J(R)$ (i.e., $J(R)$ is a radical ring; see Definition IX.2.9), which we do in the following Theorem IX.2.16(iii). These results combine to give that property P of left quasi-regularity is a radical property and its P-radical is the Jacobson radical $J(R)$.

Theorem IX.2.14. If R is a ring, then the quotient ring $R/J(R)$ is semisimple.

Lemma IX.2.15. Let R be a ring and $a \in R$.

(i) If $-a^2$ is left quasi-regular, then so is a.

(ii) $a \in J(R)$ if and only if Ra is a left quasi-regular left ideal.

Theorem IX.2.16.

(i) If an ideal I of a ring R is itself considered as a ring, then $J(I) = I \cap J(R)$.

(ii) If R is semisimple, then so is every ideal of R.

(iii) $J(R)$ is a radical ring.

Notes. We conclude this section with a result relating the Jacobson radical to direct products.

Theorem IX.2.17. If $\{R_i \mid i \in I\}$ is a family of rings, then $J\left(\prod_{i\in I} R_i\right) = \prod_{i\in I} J(R_i)$.

Revised: 10/6/2018