Section IX.3. Semisimple Rings

Note. Recall that a ring R is defined to be (Jacobson) semisimple if the Jacobson radical is zero, $J(R) = \{0\}$ (Definition IX.2.9). We give a classification of semisimple rings in Proposition IX.2.1. We give a version of the Wedderburn-Artin Theorem for semisimple Artinian rings in Theorem IX.3.3 and then give several other characterizations of such rings.

Definition IX.3.1. A ring is a *subdirect product* of the family of rings $\{R_i \mid i \in I\}$ if R is a subring of the direct product
$rod_{i \in I} R_i$ such that $\pi_k(R) = R_k$ for every $k \in I$ where $\pi_k : \prod_{i \in I} R_i \to R_k$ is the canonical epimorphism.

Definition IX.3.A. A ring S is *isomorphic to a subdirect product* of the family of rings $\{R_i \mid i \in I\}$ if there is a monomorphism (one to one homomorphism) of rings $\varphi : S \to \prod_{i \in I} R_i$ such that $\pi_k \varphi(S) = R_k$ for every $i \in I$.

Note. The idea of a subdirect product was introduced by Garret Birkhoff (January 19, 1911 – November 22, 1996) (famed coauthor of *A Survey of Modern Algebra* [1941] and *Algebra* [1967] with Saunders MacLane) in "Subdirect Unions in Universal Algebra," *Bulletin of the American Mathematical Society*, **50**(10): 764–768 (see the Wikipedia page for "Subdirect Product" at: `https://en.m.wikipedia.org/wiki/Subdirect_product`; accessed 10/7/2018). We limit our exploration of subdirect products to the next result (Proposition IX.3.2) and Exercise IX.3.2 in which the concept of subdirect irreducibility is introduced; Exercise IX.3.2(b) asks for a proof that *every ring* is isomorphic to a subdirect product of a family of subdirectly irreducible rings (a result which Hungerford credits to Birkhoff). Some authors actually define a subdirect product as what Hungerford labels "isomorphic to a subdirect product"; see page 52 of I.N. Herstein's *Noncommutative Rings*, Carus Mathematical Monographs 15, Mathematical Association of America, 1968.

Example. Let $P = \{2, 3, 5, 7, 11, \ldots\}$ be the set of prime integers. For each $k \in \mathbb{Z}$ and $p \in P$, let $k_p = \overline{k} = \{\ldots, k - 2p, k - p, k, k + p, k + 2p, \ldots\} \in \mathbb{Z}_p$ be the image of k under the canonical epimorphism mapping $\mathbb{Z} \to \mathbb{Z}_p$. Then define $\varphi : \mathbb{Z} \to \prod_{p\in P} \mathbb{Z}_p$ as $k \mapsto \{k_p\}_{p\in P}$. Then φ is a monomorphism of rings (by the Fundamental Theorem of Arithmetic). Also, for fixed $p' \in P$ and $k \in \mathbb{Z}$ we have

$$\pi_{p'}\varphi(k) = \pi_{p'}\left(\{k_p\}_{p\in P}\right) = k_{p'} = \{\ldots, k - 2p', k - p', k, k + p', k + 2p', \ldots\} \in \mathbb{Z}_{p'}.$$

So as k ranges over $\mathbb{Z}$, each equivalence class of $\mathbb{Z}_{p'}$ results from $\pi_{p'}\varphi(k)$ so that $\pi_{p'}\varphi(\mathbb{Z}) = \mathbb{Z}_{p'}$. This holds for all $p' \in P$ so that can conclude that $\pi_p\varphi(\mathbb{Z}) = \mathbb{Z}_p$ for every $p \in P$. So by Definition IX.3.A, $\mathbb{Z}$ is isomorphic to a subdirect product of the family of rings (fields, in fact) $\{\mathbb{Z}_p \mid p \in P\}$.

Note. We saw in an example of section IX.2 (see the second example after the definition of "semisimple," Definition IX.2.9) that $\mathbb{Z}$ is semisimple. This fact, along with the observation that $\mathbb{Z}$ is isomorphic to a subdirect product in the previous example, is no coincidence, as we see in the next result which gives a classification of semisimple rings.

Theorem IX.3.2. A nonzero ring R is semisimple if and only if R is isomorphic to a subdirect product or primitive rings.

Note. Just as we gave equivalent conditions to the property of "simple left Artinian ring" in the Wedderburn-Artin Theorem for Simple Artinian Rings (Theorem IX.1.14), we now give conditions equivalent to the condition "semisimple left Artinian ring."

Theorem IX.3.3. The Wedderburn-Artin Theorem for Semisimple Artinian Rings.
The following conditions on a ring R are equivalent.

(i) R is a nonzero semisimple left Artinian ring;

(ii) R is a direct product of a finite number of simple ideals each of which is isomorphic to the endomorphism ring of a finite dimensional vector space over a division ring;

(iii) there exist division rings $D_1, D_2, \ldots, D_t$ and $n_1, n_2, \ldots, n_t \in \mathbb{N}$ such that R is isomorphic to the ring $\mathrm{Mat}_{n_1}(D_1) \times \mathrm{Mat}_{n_2}(D_2) \times \cdots \times \mathrm{Mat}_{n_t}(D_t)$.

Note. We now extract some more properties of semisimple left Artinian rings from the Wedderburn-Artin Theorem for Semisimple Artinian Rings (Theorem IX.3.3).

Corollary IX.3.4.

(i) A semisimple left Artinian ring has an identity.

(ii) A semisimple ring is left Artinian if and only if it is right Artinian.

(iii) A semisimple left Artinian ring is both left and right Noetherian.

Note. A result related to Corollary IX.3.4, but not involving semisimple rings is given in Exercise IX.3.13 where it is to be shown that a left Artinian ring with identity is Noetherian.

Note. The next result is not needed in "the sequel," but it gives a description of ideals in a semisimple left Artinian ring.

Corollary IX.3.5. If I is an ideal in a semisimple left Artinian ring R, then $I = Re$, where e is an idempotent element (that is, $e^2 = e$) which is in the center of R.

Note. The Wedderburn-Artin Theorem for Semisimple Artinian Rings (Theorem IX.3.3) characterizes semisimple Artinioan rings in "ring theoretic terms." We next classify semisimple Artinian ring R in terms of properties of R-modules. This will ultimately allow us to give a definition of "semisimple ring R "in terms of R-modules, showing our definition of semisimple is consistent with that of other sources.

Theorem IX.3.6. The following conditions on a nonzero module A over a ring R are equivalent.

(i) A is the sum of a family of simple modules.

(ii) A is the (internal) direct sum of a family of simple submodules.

(iii) For every nonzero element a of A, we have $Ra \neq \{0\}$; and every submodule B of A is a direct summand of A (that is, $A = B \oplus C$ for some submodule C of A).

Definition IX.3.B. A module that satisfies the equivalent conditions of Theorem IX.3.6 is *simple* (or *completely reducible*).

Note. Conditions (i) and (ii) of Theorem IX.3.6 for semisimple *modules* is similar to condition (ii) of Theorem IX.3.3 for semisimple *rings*. Also, in Theorem IX.3.7(v), if R is a semisimple left Artinian ring with identity, then every nonzero unitary left R-module is semisimple. These facts (according to Hungerford, page 437) are the motivation for the terminology "semisimple module."

Note. We now give a number of conditions, in terms of R-modules, equivalent to the condition of ring with identity R being semisimple (left) Artinian (by Corollary IX.3.4(i), every semisimple Artinian ring has an identity so the existence of an identity is not an added constraint). First we need a definition (the following idea surfaced in the proof of Theorem IX.3.5).

Definition. A subset $\{e_1, e_2, \dots, e_m\}$ of ring R is a set of *orthogonal idempotents* if $e_i^2 = e_i$ for all i and $e_i e_j = 0$ for all $i \neq j$.

Theorem IX.3.7. The following conditions on a nonzero ring R with identity are equivalent.

(i) R is semisimple left Artinian;

(ii) every unitary left R-module is projective;

(iii) every unitary left R-module is injective;

(iv) every short exact sequence of unitary left R-modules is split exact;

(v) every nonzero unitary left R-module is semisimple;

(vi) R is itself a unitary semisimple left R-module;

(vii) every left ideal of R is of the form Re with e idempotent;

(viii) R is the (internal) direct sum (as a left R-module) of minimal left ideals $K_1, K_2, \dots, K_m$ such that $K_i = Re_i$ (where $e_i \in K_i$) for $i = 1, 2, \dots, m$ and $\{e_1, e_2, \dots, e_m\}$ is a set of orthogonal idempotents with $e_1 + e_2 + \cdots + e_m = 1_R$.

Note. Since a semisimple ring is left Artinian if and only if it is right Artinian by Corollary IX.3.4(ii), we can replace "left" with "right" in any (or all) parts of Theorem IX.3.7. If we remove the word "unitary" from parts (ii), (iii), (iv), (v), or (vi) then the result does not hold, as is to be shown in Exercise IX.3.10. The results of this section to this stage give decompositions of semisimple left Artinian rings; we now show the uniqueness of the decomposition into simple ideals (as given in Theorem IX.3.3(ii)).

Proposition IX.3.8. Let R be a semisimple left Artinian ring.

(i) $R = I_1 \times I_2 \times \cdots \times I_n$ where each I_i is a simple ideal of R.

(ii) If J is any simple ideal of R, then $J = I_k$ for some k.

(iii) If $R = J_1 \times J_2 \times \cdots \times J_m$ with each J_k a simple ideal of R, then $n = m$ and (after re-indexing) $I_k = J_k$ for $k = 1, 2, \ldots, n$.

Definition. The uniquely determined simple ideals $I_1, I_2, \ldots, I_n$ in Proposition IX.3.8 are the *simple components* of semisimple left Artinian ring.

Note. The uniqueness of the decomposition of a semisimple left Artinian ring into minimal left ideals (as given in Theorem IX.3.7(iii)) is implied by the following.

Proposition IX.3.9. Let A be a semisimple module over a ring R. If there are direct sum decompositions $A = B_1 \oplus B_2 \oplus \cdots \oplus B_m$ and $A = C_1 \oplus C_2 \oplus \cdots \oplus C_n$, where each B_i and C_j is a simple module of A, then $m - n$ and (after re-indexing) $B_i \cong C_i$ for $i = 1, 2, \ldots, m$.

Note. In Proposition IX.3.8 we have *equality* of the simple ideals, but in Proposition IX.3.9 we only have isomorphism of the simple submodules; Proposition IX.3.9 is false if we replace the isomorphism claim with an equality claim, as is to be shown in Exercise IX.3.11.

Note. The nest theorem is used in the proof of Theorem IX.6.7, which has as a corollary Frobenius' Theorem (which states that an algebraic division algebra over $\mathbb{R}$ is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ [the noncommutative division ring of quaternions]).

Theorem IX.3.10. Let R be a semisimple left Artinian ring.

(i) Every simple left (respectively, right) R-module is isomorphic to a minimal left (right) ideal of R.

(ii) The number of nonisomorphic simple left (respectively, right) R-modules is the same as the number of simple components of R.

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