Quaternions—The Degree of a Polynomial versus the Number of Zeros

Note. In Thomas Hungerford's *Algebra* (Springer-Verlag, 1974), the real quaternions are defined as follows (see page 117): Let $S = \{1, i, j, k\}$. Let $\mathbb{H}$ be the additive abelian group $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ and write the elements of $\mathbb{H}$ as formal sums $(a_0, a_1, a_2, a_3) = a_0 1 + a_1 i + a_2 j + a_3 k$. We often drop the "1" in "$a_0 1$" and replace it with just a_0. Addition in $\mathbb{H}$ is as expected:

$$(a_0+a_1i+a_2j+a_3k)+(b_0+b_1i+b_2j+b_3k) = (a_0+b_0)+(a_1+b_1)i+(a_2+b_2)j+(a_3+b_3)k.$$

We turn $\mathbb{H}$ into a ring by defining multiplication as

$$(a_0 + a_1 i + a_2 j + a_3 k)(b_0 + b_1 i + b_2 j + b_3 k) = (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3)$$

$$+(a_0b_1+a_1b_0+a_2b_3-a_3b_2)i+(a_0b_2+a_2b_0+a_3b_1-a_1b_3)j+(a_0b_3+a_3b_0+a_1b_2-a_2b_1)k.$$

This product can be interpreted by considering:

(i) multiplication in the formal sum is associative,

(ii) $ri = ir$, $rj = jr$, $rk = kr$ for all $r \in \mathbb{R}$,

(iii) $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

This ring is called the *real quaternions* (or simply the "quaternions").

Note. From (iii) above we see that $ij = k$ and $ji = -k$, so that $ij \neq ji$ and $\mathbb{H}$ is not a commutative ring. Recall that a ring D with identity $1_D \neq 0$ in which every nonzero element is a unit (i.e., has a multiplicative inverse) is an *integral domain.*

The quaternions are the standard example of a noncommutative division ring (see Theorem A of Supplement. Quaternions—An Algebraic View).

Note. In Supplement. Quaternions—An Algebraic View, we considered the structure of the set of zeros of a quaternionic polynomial and saw that it is much more complicated than in the settings of the real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$. It is shown that $q^2+1 \in \mathbb{H}[x]$ has, not two, but uncountably many zeros! In Note [???] of Supplement. Quaternions—An Algebraic View it is shown that $x_1 i + x_2 j + x_3 k$ is a zero of q^2+1 if x_1, x_2, x_3 satisfy $x_1^2 + x_2^2 + x_3^2 = 1$. This inspired the following definition.

Definition. We denote by $\mathbb{S}$ the two dimensional sphere (as a subset of the four dimensional quaternions $\mathbb{H}$) $\mathbb{S} = \{q = x_1 i + x_2 j + x_3 k \mid x_1^2 + x_2^2 + x_3^2 = 1\}$. As observed above, for any $I \in \mathbb{S}$ we have $I^2 = -1$. For $x, y \in \mathbb{R}$ we let $x + y\mathbb{S}$ denote the two dimensional sphere $x + y\mathbb{S} = \{x + yI \mid I \in \mathbb{S}\}$. (We might think of $x + y\mathbb{S}$ as a two dimensional sphere centered at $(x, 0, 0, 0)$ with radius $|y|$.)

Note. Most of the remainder of this supplement is based on the following:

1. A. Pogorui and M. Shapiro, On the Structure of the Set of Zeros of Quaternionic Polynomials, *Complex Variables*, **49**(6) (2004), 379–389.

2. G. Gentili and C. Stoppato, Zeros of Regular Functions and Polynomials of a Quaternionic Variable, *Michigan Mathematics Journal* **56** (2008), 655–667.

3. G. Gentili and D. Struppa, On the Multiplicity of Zeroes of Polynomials with Quaternionic Coefficients, *Milan Journal of Mathematics* bf 76 (2008), 15–25.

Note. The next result is fundamental in the structure of the set of zeros of a quaternionic polynomial. It originally appears in Pogorui and Shapiro (2004). A proof is given in Supplement. Quaternions—An Algebraic View (see the proof of Theorem B).

Theorem B. Let $p(q) = \sum_{n=0}^{N} q^n a_n$ be a given quaternionic polynomial. Suppose that there exist $x_0, y_0 \in \mathbb{R}$ and $I, J \in \mathbb{S}$ with $I \neq J$ such that $p(x_0 + y_0 I) = 0$ and $p(x_0 + y_0 J) = 0$. Then for all $L \in \mathbb{S}$ we have $p(x_0 + y_0 L) = 0$.