Supplement. Direct Products and Semidirect Products

Note. In Section I.8 of Hungerford, we defined direct products and weak direct products. Recall that when dealing with a finite collection of groups $\{G_i\}_{i=1}^n$ then the direct product and weak direct product coincide (Hungerford, page 60). In this supplement we give results concerning recognizing when a group is a direct product of smaller groups. We also define the semidirect product and illustrate its use in classifying groups of small order. The content of this supplement is based on Sections 5.4 and 5.5 of Davis S. Dummitt and Richard M. Foote's *Abstract Algebra*, 3rd Edition, John Wiley and Sons (2004).

Note. Finitely generated abelian groups are classified in the Fundamental Theorem of Finitely Generated Abelian Groups (Theorem II.2.1). So when addressing direct products, we are mostly concerned with nonabelian groups. Notice that the following definition is "dull" if applied to an abelian group.

Definition. Let G be a group, let $x, y \in G$, and let A, B be nonempty subsets of G.

(1) Define $[x, y] = x^{-1}y^{-1}xy$. This is the *commutator* of x and y.

(2) Define $[A, B] = \langle [a, b] \mid a \in A, b \in B \rangle$, the group generated by the commutators of elements from A and B where the binary operation is the same as that of group G.

(3) Define $G' = \{[x, y] \mid x, y \in G\rangle$, the subgroup of G generated by the commutators of elements from G under the same binary operation of G. This is called the *commutator subgroup* of G.

Note. We have that $[x, y] = 1$ (we are using multiplicative notation here) if and only if $xy = yx$. If G is abelian then $G' = \langle 1 \rangle$ (and conversely).

Proposition DF.5.7. Let G be a group, let $x, y \in G$, and let $H \leq G$. Then

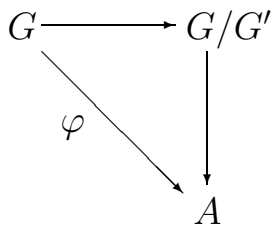
(1) $xy = yx[x, y]$.

(2) $H \trianglelefteq G$ if and only if $[H, G] \leq H$.

(3) For any automorphism σ of G, we have $\sigma([x, y]) = [\sigma(x), \sigma(y)]$. Also, G' is a *characteristic subgroup* of G (denoted "G' char G"; this means that every automorphism of G maps G' to itself, i.e., $\sigma(G') = G'$) and G/G' is abelian.

(4) G/G' is the largest abelian quotient group of G in the sense that if $H \trianglelefteq G$ and G/H is abelian, then $G' \leq H$. Conversely, if $G' \leq H$, then $H \trianglelefteq G$ and G/H is abelian.

(5) If $\varphi : G \to A$ is any homomorphism of G into an abelian group A, then φ factors through G', i.e., $G' \leq \ker(\varphi)$ and the following diagram commutes:



Note. Another way of viewing (4) of Proposition DF.5.7 is to notice that a quotient G/H (which automatically assumes $H \trianglelefteq G$) is abelian if and only if the commutator subgroup G' is a subgroup of G, $G' \leq H$. Or, G/G' is the largest abelian quotient of G because g' is the smallest subgroup yielding an abelian quotient.

Note. Dummitt and Foote call the following result the "Recognition Theorem." It is Hungerford's Corollary I.8.7 for a collection of two groups (as opposed to a finite collection, as given by Hungerford).

Theorem DF.5.9. Recognition Theorem for Direct Products.
Suppose G is a group with subgroups H and K such that

(1) H and K are normal in G, and

(2) $H \cap K = \{1\}$.

Then $HK = H \times K$.

Note. We now redefine "direct products," now for just a product of two groups.

Definition. If G is a group and H and K are normal subgroups of G with $H \cap K = \{1\}$, we call $HK = \{hk \mid h \in H, k \in K\}$ the *internal direct product* of H and K. We call $H \times K = \{(h, k) \mid h \in H, k \in K\}$ the *external direct product* of H and K.

Note. By the Recognition Theorem (Theorem DF.5.9), HK and $H \times K$ are isomorphic, hence we simply speak of the "direct product" of H and K.

Example. If n is a positive odd integer, then we claim $D_{2n} \cong D_n \times \mathbb{Z}_2$. Let $D_{2n} = (\{r, s\} \mid \{r^{2n} = 1, s^2 = 1, srs = r^{-1}\})$ (see Exercise I.9.8 of Hungerford). Let $H = \langle s, r^2 \rangle$ and let $K = \langle r^n \rangle$. The geometric interpretation is that D_{2n} is the group of symmetries of a regular $2n$-gon, H is the group of symmetries of the regular n-gon inscribed in the $2n$-gon based on every other vertex of the $2n$-gon (say those with an even label), and K is the group of symmetries of the $2n$-gon which rotates the $2n$-gon half way around (notice $(r^n)^2 = r^{2n} = 1$). Since $[D_{2n} : H] = 2$, then $H \trianglelefteq D_{2n}$. Since $srs = r^{-1}$, $s^2 = 1$, and $s = s^{-1}$, then we have $(srs)^n = sr^n s = (r^{-1})^n = r^{-n} = r^n$. That is, s "centralizes" r^n (recall that the centralizer of x in G is $C_G(x) = \{g \in G \mid gxg^{-1} = x\}$). Since $rr^n r^{-1} = r^n$ then r also centralizes r^n. The *center* of group G is

$$\{g \in G \mid gx = xg \text{ for all } x \in G\} = \{g \in G \mid gxg^{-1} = x \text{ for all } x \in G\}$$

denoted $C(G)$ by Hungerford and denoted $Z(G)$ by Dummitt and Foote. So we have $\langle r^n \rangle = K \leq Z(D_{2n}) = C(D_{2n})$. So $\langle r^n \rangle = K \trianglelefteq D_{2n}$ because $D_{2n} = \langle r, s \rangle$ and both r and s centralize r^n. Finally, $K \not\leq H$ since $|K| = 2$ (because $(r^n)^2 = 1$) and $H = \langle s, r^2 \rangle$ where the order of r^2 is n (and n is odd—this is where the oddness of n is used); that is, $r^n \notin \langle r^2 \rangle \leq \langle s, r^2 \rangle$. Since $H \not\leq K$ then $H \cap K = \{1\}$ by Lagrange's Theorem (Corollary I.4.6). By the Recognition Theorem (Theorem DF.5.9), $HK \cong H \times K \cong D_n \times \mathbb{Z}_2$. Since $|H \times K| = 4n$, then $|HK| = 4n$ and since $HK \leq D_{2n}$, then it must be that $HK = D_{2n}$. So $D_{2n} = HK \cong H \times K \cong D_n \times \mathbb{Z}_2$.

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Note. We now turn our attention from direct products to "semidirect products." Recall that the internal direct product of H and K (subgroups of a group G), HK, required that both H and K be normal subgroups of group G. In defining the semidirect product, we relax the normality condition but introduce a use of automorphisms of one of the two groups. We will be able to construct non-abelian groups from H and K, even if both H and K are abelian.

Note. Exercise I.4.6 of Hungerford (and Proposition 3.14 of Dummitt and Foote) states: If H and K are subgroups of a group, then HK is a subgroup if and only if $HK = KH$.

Corollary DF.3.15. If H and K are subgroups of G and $H \leq N_G(K) = \{g \in G \mid gKg^{-1} = K\}$, then HK is a subgroup of G. In particular, if $K \trianglelefteq G$ then $HK \leq G$ for any $H \leq G$.

Note. As motivation, suppose G is a group $H \leq G$, $K \leq G$, $H \trianglelefteq G$, and $H \cap K = \{1\}$. Then HK is a subgroup of G by Corollary DF.3.15 and every element of HK can be uniquely written as a product hk for $h \in H$, $k \in K$. For $h_1k_1, h_2k_2 \in HK$ we have in group G that

$$(h_1k_1)(h_2k_2) = h_1k_1h_2(k_1^{-1}k_1)k_2 = h_1(k_1h_2k_1^{-1})k_1k_2 = h_3k_3 \in HK$$

where $h_3 = h_1(k_1h_2k_1^{-1}) \in H$ (since $H \trianglelefteq G$ and so $k_1h_2k_1^{-1} \in H$) and $k_3 = k_1k_2$. Notice that all this depends on starting with a group G that has H, K as subgroups. To define the semidirect product, we want to liberate ourselves from this setting and find a way to interpret the quantity $k_1h_2k_2^{-1}$ independent of group G.

Note. If we let group K act on group H by conjugation, that is $k \cdot h = khk^{-1}$ for $h \in H$, $k \in K$ where $\cdot$ represents action, then

$$(h_1k_1)(h_2k_2) = (h_1k_1h_2h_1^{-1})(h_1h_2) = (h_1k_1 \cdot h_2)(k_1k_2).$$

Now action of k on H by conjugation yields an automorphism of H, $kHk^{-1} = H$ since $H \trianglelefteq G$, for each $k \in K$. So we can define a homomorphism φ of K into $\text{Aut}(H)$ where $\varphi(k)$ is the automorphism of H created from the mapping $h \to khk^{-1}$ for all $h \in H$. So the quantity $(h_1k_1 \cdot h_1)(k_1k_2)$ depends only on the binary operation in H, the binary operation in K, and the homomorphism φ from K to $\text{Aut}(H)$. Notice that for such a general φ (not necessarily conjugation) we have $k_1 \cdot h_2 \in H$ since $\varphi(k_1) \in \text{Aut}(H)$ and so k_1 acts on h_2 by mapping h_2 to another element of H under the automorphism $\varphi(k_1)$. Strictly speaking, we have ordered pairs of elements, the first an element of H and the second an element of K. The following results shows that this approach works and that we only need groups H, K, and φ a homomorphism mapping K to $\text{Aut}(H)$.

Theorem DF.5.10. Let H and K be groups and let φ be a homomorphism from K into $\text{Aut}(H)$. Let $\cdot$ denote action of K on H determined by φ. Let G be the set of ordered pairs (h, k) with $h \in H$ and $k \in K$ and define the binary operation $(h_1, k_1)(h_2, k_2) = (h_1k_1 \cdot h_2, k_1, k_2)$.

(1) The binary operation makes G a group of order $|G| = |H||K|$.

(2) The sets $\tilde{H} = \{(h, 1) \mid h \in H\}$ and $\tilde{K} = \{(1, k) \mid k \in K\}$ are subgroups of G and the maps $h \mapsto (h, 1)$ for $h \in H$ and $k \mapsto (1, k)$ for $k \in K$ are isomorphisms of these subgroups with groups H and K.

(3) $H \trianglelefteq G$ (associating H with its isomorphic copy of ordered pairs).

(4) $H \cap K = \{1\}$.

(5) For all $h \in H$ and $k \in K$, we have $khk^{-1} = k \cdot h = \varphi(k)(h)$.

Definition. Let H and K be groups and let φ be a homomorphism from K into $\text{Aut}(H)$. The group G described in Theorem DF.5.10 is the *semidirect product* of H and K with respect to φ. This is denoted $G = H \rtimes_\varphi K$, or simply $G = H \rtimes K$ if there is no confusion as to what φ is.

Note. The notation "$\rtimes$" is meant to reflect the fact that H is a normal subgroup of G (by Theorem DF.5.10(3))—the right hand half of the symbol is like the "$\triangleleft$" in the normal subgroup notation.

Proposition DF.5.11. Let H and K be groups and let $\varphi : K \to \text{Aut}(H)$ be a homomorphism. The following are equivalent.

(1) The identity set map between $H \rtimes K$ and $H \times K$ (both consisting of ordered pairs) is a group homomorphism (and hence $H \rtimes K \cong H \times K$).

(2) φ is the trivial homomorphism from K into $\text{Aut}(H)$ (which maps all $k \in K$ to the identity automorphism).

(3) $K \trianglelefteq H \rtimes K$.

Example A. Let H be any abelian group and let $K = \langle x \rangle \cong \mathbb{Z}_2$. Define homomorphism $\varphi : K \to \mathrm{Aut}(H)$ by defining the action of $x \in K$ on $h \in H$ as $x \cdot h = h^{-1}$. Then $G = H \rtimes_\varphi K$ contains the subgroup H (technically $\tilde{H}$ in the notation of Theorem DF.5.10(2)) of index 2 (since there are 2 cosets of H, namely $1H = H$ and xH) and so by Theorem DF.5.10(5),

$$x \cdot h = xhx^{-1} = h^{-1} \text{ for all } h \in H.$$

If $H \cong \mathbb{Z}_n$ then $G = H \rtimes_\varphi K \cong D_n$ (think of elements of $\mathbb{Z}_n$ as rotations; the action of $K \cong \mathbb{Z}_2$ on H is like a mirror image combined with rotations). In other words, $D_n \cong \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$. To see this more clearly, recall that a presentation of D_n is given by $(\{r, m\} \mid \{r^n = m^2 = 1,\, rm = mr^{-1}\})$. Since we have $xhx^{-1} = h^1$ for all $h \in H$ then $x^2hx^{-1} = xh^{-1}$ or, since $x^2 = 1$ and $x^{-1} = x$, $hx = xh^{-1}$. So the isomorphism between D_n and $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ is given by mapping $x \mapsto m$ and $h \mapsto r$ where h is a generator of $\mathbb{Z}_n$. Similarly, $D_\infty \cong \mathbb{Z} \rtimes_\varphi \mathbb{Z}_2$.

Example B. We modify the previous example. We let H be any abelian group and let $K = \langle x \rangle \cong \mathbb{Z}_{2n}$. Define K acting on H again by $x \cdot h = h^{-1}$ (so again x^2 acts as the identity on H), and again $xhx^{-1} = h^{-1}$ for all $h \in H$. Also, $x^2hx^{-2} = h$ since x^2 and x^{-2} act as the identity. Hence $x^2h = hx^2$ for all $h \in H$. So x^2 commutes with all elements of H and, since $\mathbb{Z}_{2n}$ is cyclic, x^2 commutes with all elements of K. So $x^2 \in C(H \rtimes_\varphi K) = \{g \in H \rtimes_\varphi K \mid g\ell = \ell g \text{ for all } \ell \in H \rtimes_\varphi K\}$ (called the *center* of $H \rtimes_\varphi K$). In this example, we denote the identity of H and K both as e (instead of 1, since we now consider specific additive groups). Notice that for $(h, e), (e, x) \in H \rtimes_\varphi \mathbb{Z}_{2n}$ we have $(h, e)(e, x) = (h\, e \cdot e, ex) = (h, x)$ and

$(e,x)(h,e) = (h\,e \cdot h, xe) = (x \cdot h, x) = (h^{-1}, x)$. So $H \rtimes_\varphi \mathbb{Z}_{2n}$ is nonabelian if $h \neq h^{-1}$ for some $h \in H$. In particular, for $H = \mathbb{Z}_3$ and $K = \mathbb{Z}_4$, we see that $H \rtimes_\varphi K = \mathbb{Z}_3 \rtimes_\varphi \mathbb{Z}_4$ is a nonabelian group of order 12 (by Theorem DF.10(1)). We already know that A_4 and D_{12} are nonabelian groups of order 12. Now $\mathbb{Z}_3 \rtimes_\varphi \mathbb{Z}_4$ has $\mathbb{Z}_4$ as a subgroup (in fact, this is a Sylow p-subgroup). However, the Sylow p-subgroups (for $p = 2$) of A_4 and D_4 are $V = \mathbb{Z}_2 \times \mathbb{Z}_2$ (the Klein-4 group). So $\mathbb{Z}_3 \rtimes_\varphi \mathbb{Z}_4$ is isomorphic to neither D_4 nor A_4. We mentioned in the notes for Hungerford's Section II.6, that a presentation for the dicyclic group Dic_3 (or order 12) is $(\{a,b\}, \{a^6 = e, a^3 = b^2, ab = ba^{-1}\})$. Let y be a generator of $H \cong \mathbb{Z}_3$ and let x be a generator of $K \cong \mathbb{Z}_4$. Define $a = (y, x^2)$ and $b = (e, x)$. Then $b^n = (e, x^n)$ and so $b^4 = (e, e)$. Also,

$$a^3 = (y,x^2)^3 = (y,x^2)(y\,x^2 \cdot y, x^4) = (y,x^2)(y^2,e)$$

$$= (y\,x^2 \cdot y^2, x^2) = (y^3, x^2) = (e, x^2) = b^2,$$

and so $a^6 = (e,e)$. Now $a^{-1} = (x^{-2} \cdot y^{-1}, x^{-2}) = (y^{-1}x^2)$ and so

$$ab = (y,x^2)(e,x) = (y\,x^2 \cdot e, x^3) = (y, x^3),$$

$$ba^{-1} = (e,x)(y^{-1},x^2) = (e\,x \cdot y^{-1}, x^3) = (y, x^3).$$

So the presentation of Dic_3 is satisfied by these two elements of $\mathbb{Z}_3 \rtimes_\varphi \mathbb{Z}_4$. Since Dic_3 and $\mathbb{Z}_3 \rtimes_\varphi \mathbb{Z}_4$ are both of order 12, then in fact $\mathrm{Dic}_3 \cong \mathbb{Z}_3 \rtimes_\varphi \mathbb{Z}_4$. This doesn't change our list of groups of order 12 given in Hungerford's Section II.6, but it does give us an alternate interpretation of Dic_3 in terms of semidirect products.

We now give another representation of this group. Based on the notation used in *Schaum's Outline of Theory and Problems of Group Theory* by B. Baumslag and

B. Chandler (1968), we can let $A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $B = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}$ where $i^2 = -1$ and $\epsilon^3 = 1$ but $\epsilon \neq 1$ (so ϵ is a primitive complex cube root of 1). Then we get the following multiplication table.

	1	A	A^2	A^3	B	B^2	AB	A^2B	A^3B	AB^2	A^2B^2	A^3B^2
1	1	A	A^2	A^3	B	B^2	AB	A^2B	A^3B	AB^2	A^2B^2	A^3B^2
A	A	A^2	A^3	1	AB	AB^2	A^2B	A^3B	B	A^2B^2	A^3B^2	B^2
A^2	A^2	A^3	1	A	A^2B	A^2B^2	A^3B	B	AB	A^3B^2	B^2	AB^2
A^3	A^3	1	A	A^2	A^3B	A^3B^2	B	AB	A^2B	B^2	AB^2	A^2B^2
B	B	AB^2	A^2B	A^3B^2	B^2	1	A	A^2B^2	A^3	AB	A^2	A^3B
B^2	B^2	AB	A^2B^2	A^3B	1	B	AB^2	A^2	A^3B^2	A	A^2B	A^3
AB	AB	A^2B^2	A^3B	B^2	AB^2	A	A^2	A^3B^2	1	A^2B	A^3	B
A^2B	A^2B	A^3B^2	B	AB^2	A^2B^2	A^2	A^3	B^2	A	A^3B	1	AB
A^3B	A^3B	B^2	AB	A^2B^2	A^3B^2	A^3	1	AB^2	A^2	B	A	A^2B
AB^2	AB^2	A^2B	A^3B^2	B	A	AB	A^2B^2	A^3	B^2	A^2	A^3B	1
A^2B^2	A^2B^2	A^3B	B^2	AB	A^2	A^2B	A^3B^2	1	AB^2	A^3	B	A
A^3B^2	A^3B^2	B	AB^2	A^2B	A^3	A^3B	B^2	A	A^2B^2	1	AB	A^2

If we associate matrix A with element b above and associate matrix A^2B with element a above (or if we associate matrices A and B with elements $(1, x)$ and $(y, 1)$ in $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$, respectively), then this yields an isomorphism between the group generated by matrices A and B and $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$.

Example C. Let p and q be primes such that $p > q$. By Hungerford's Theorem II.6.1, if $q \nmid (p-1)$ then every group of order pq is isomorphic to $\mathbb{Z}_{pq}$. If $q \mid (p-1)$ then there is (up to isomorphism) one abelian group $\mathbb{Z}_{pq}$ of order pq and one nonabelian group of order pq. We now show that, in fact, the nonabelian group is $G = \mathbb{Z}_p \rtimes \mathbb{Z}_q$ for some $\varphi \in \mathrm{Aut}(\mathbb{Z}_q)$. Notice that when $q = 2$, the nonabelian group of order $2q$ is isomorphic to D_p by Hungerford Corollary II.6.2 (and this is isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_2$ by Example A). So now assume $q \mid (p-1)$. Then $H = \mathbb{Z}_p$ is cyclic and has $p-1$ different generators (namely, $1, 2, \ldots, p-1$). So $\mathrm{Aut}(\mathbb{Z}_p)$ is a group of order $p-1$ (every automorphism is determined by where, say 1, is mapped since 1 generates $\mathbb{Z}_p$, and the image of 1 generates $\mathbb{Z}_p$—so there are $p-1$ choices for the image of 1). Since q is prime and $q \mid (p-1)$, then by Cauchy's Theorem (Hungerford's Theorem II.5.2) $\mathrm{Aut}(\mathbb{Z}_p)$ has a subgroup of order q. Since this subgroup is order q (prime) then it is cyclic, and $K = \mathbb{Z}_q$ is cyclic of order q, so there is an isomorphism from K to this subgroup of $\mathrm{Aut}(\mathbb{Z}_p)$. In other words, there is a homomorphism φ from $K = \mathbb{Z}_q$ to $\mathrm{Aut}(H) = \mathrm{Aut}(\mathbb{Z}_p)$. Notice that φ is a nontrivial homomorphism (since its image is not the identity automorphism of $\mathbb{Z}_p$). So we can form the group $H \rtimes K = \mathbb{Z}_p \rtimes \mathbb{Z}_q$. By Proposition DF.5.11, $K = \mathbb{Z}_q$ is not a normal subgroup of $H \rtimes K$. Therefore $H \rtimes K$ is not abelian (for if it were, all subgroups would be normal). So the unique nonabelian group of order pq is isomorphic to $\mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_q$ where φ is as described above.

Note. Dummit and Foote state a "Recognition Theorem" for semidirect products. As with the Recognition Theorem for Direct Products, this new Recognition Theorem tells us when $G = HK$ is isomorphic to $H \rtimes K$.

Theorem DF.5.12. Recognition Theorem for Semidirect Products.

Suppose G is a group with subgroups H and K such that

(1) $H \trianglelefteq G$, and

(2) $H \cap K = \{1\}$.

Let $\varphi : K \to \text{Aut}(H)$ be the homomorphism defined by mapping $k \in K$ to the automorphism of left conjugation by k on H. Then $HK \cong H \rtimes K$. In particular, if $G = HK$ with H and K satisfying (1) and (2), then G is the semidirect product of H and K.

Note. To use the Recognition Theorem for Semidirect Products (Theorem DF.5.12), we follow the outline:

(a) Show that every group of order n has proper subgroups H and K such that $H \trianglelefteq G$, $H \cap K = \{1\}$, and $G = HK$.

(b) Find all possible groups isomorphic to H or K.

(c) For each pair H, K in (b). find all possible homomorphisms $\varphi : K \to \text{Aut}(H)$.

(d) For each H, K, φ in (c), form the semidirect product $H \rtimes K$ and determine among the resulting groups which are isomorphic.

This gives a list of the distinct isomorphic types of groups of order n. We now illustrate this approach in two examples.

Example. Groups of order 30.

First, we show that a group G of order 30 has a subgroup of order 15. Group G has Sylow p-subgroups of orders 3 and 5 by the First Sylow Theorem (Theorem II.5.7). The Sylow theorems can be used to show that either the group of order 3 or the group of order 5 is a normal subgroup of G (see Fraleigh's Example 37.12 or Dummit and Foote's example on their pages 143–144). Then by Corollary DF.3.15 (the "in particular" part) the product H_1K_1 of the two Sylow p-subgroups is a subgroup of group G and $|H_1K_1| = 15$. So G has a subgroup of order 15; this subgroup is normal in G since the index $(G : H) = 2$, and so G is not simple.

By the First Sylow Theorem (Theorem II.5.7) G has a Sylow p-subgroup of order 2, say K. Let H by the normal subgroup of order 15. The elements of H are of order 1, 3, 5, or 15. The elements of K are of order 1 or 2. So $H \cap K = \{1\}$. So H and K satisfy the hypotheses of the Recognition Theorem for Semidirect Products (Theorem DF.5.12) and so (by the "in particular" part), $G \cong H \rtimes K$ for some $\varphi : K \to \mathrm{Aut}(H)$. This completes part (a) of the outline.

Now $|H| = 3 \times 5$ and so by Example C above, we have that $H \cong \mathbb{Z}_{15}$. Since $|K| = 2$ then $K \cong \mathbb{Z}_2$ and this completes (b) of the outline.

One can show that the automorphism group of H is $\mathrm{Aut}(H) = \mathrm{Aut}(\mathbb{Z}_{15}) \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. (Proposition 4.16 of Dummit and Foote implies that $\mathrm{Aut}(\mathbb{Z}_{15})$ is isomorphic to the group of units of $\mathbb{Z}_{15}$ under multiplication, $(\mathbb{Z}_{15})^\times$, a group of order 8. It is then straightforward to show that $(\mathbb{Z}_{15})^\times \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.) So $\mathrm{Aut}(H)$ contains precisely three elements of order 2. Now $H \cong \mathbb{Z}_{15} \cong \mathbb{Z}_5 \times \mathbb{Z}_3$ so, say, $H = \langle a \rangle \times \langle b \rangle$ where a has order 5 and b has order 3. Then the three elements of $\mathrm{Aut}(H)$ of order 2 must

behave as follows:

$$\alpha_1 : \begin{array}{l} a \mapsto a \\ b \mapsto b^{-1}, \end{array} \qquad \alpha_2 : \begin{array}{l} a \mapsto a^{-1} \\ b \mapsto b, \end{array} \qquad \alpha_3 : \begin{array}{l} a \mapsto a^{-1} \\ b \mapsto b^{-1}. \end{array}$$

Since K is of order 2, then there are three nontrivial homomorphisms from K to $\mathrm{Aut}(H)$ given by sending the generator of K into one of α_1, α_2, or α_3. There is also the trivial homomorphism from K to $\mathrm{Aut}(H)$, but this yields a direct product and $HK = H \times K \cong \mathbb{Z}_{30}$ This completes (c) of the outline. We now go through the list of the three nontrivial homomorphisms to complete part (d) of the outline.

Let $K = \langle k \rangle$. If $\varphi_1 : K \to \mathrm{Aut}(H)$ is defined as $\varphi_1(k) = \alpha_1$ (so in terms of group action, $k \cdot a = \varphi_1(a) = a$ and $k \cdot b = \varphi_1(b) = b^{-1}$). Then $G_1 = H \rtimes_{\varphi_1} K$. Dummit and Foote claim that it is easy to see that $G_1 \cong \mathbb{Z}_5 \times D_3$ (Hmmm...).

If $\varphi_2 : K \to \mathrm{Aut}(H)$ is defined as $\varphi_2(k) = \alpha_2$ (so in terms of group action, $k \cdot a = \alpha_2(a) = a^{-1}$ and $k \cdot b = \alpha_2(b) = b$). Then $G_2 \cong H \rtimes_{\varphi_2} K$ and "it is easily seen" that $G_2 \cong \mathbb{Z} \times D_5$.

If $\varphi_3 : K \to \mathrm{Aut}(H)$ is defined as $\varphi_3(k) = \alpha_3$ (so in terms of group action, $k \cdot a = \alpha_3(a) = a^{-1}$ and $k \cdot b = \alpha_3(b) = b^{-1}$). Then $G_3 \cong H \rtimes_{\varphi_3} K$ and "it is easily seen" that $G_3 \cong D_{15}$.

By considering the centers of these four groups, $\mathbb{Z}_{30}$, $\mathbb{Z}_5 \times D_3$, $\mathbb{Z}_3 \times D_5$, and D_{15} (which are of sizes 30, 4, 3, and 1, respectively) we find that no pair of these groups is isomorphic, completing part (d) of the outline.

Note. The use of the Recognition Theorem for Semidirect Products was more for its use in restricting the population of groups to only four, as opposed to actually giving us a new group that could not be expressed as a direct product of groups already encountered.

Note. The idea of internal and external products can be extended to semidirect products. In A. S. Abhyankar and C. Christensen's "Semidirect Products: $x \mapsto ax + b$ as a First Example" (*Mathematics Magazine* **75**(4), 284–289 (2002)), the definitions of these two concepts are given. Dummit and Foote's definition of semidirect product (as justified by Theorem DF.5.10) is what Abhyankar and Christensen refer to as the *external semidirect product*. Dummit and Foote's Recognition Theorem for Semidirect Products (Theorem DF.5.12) describe the conditions under which a group is, in Abhyankar and Christensen's terminology, an *inner direct product*. This is consistent with Hungerford's use of the terms "internal" and "external." When speaking of G as an internal product, group G is expressed as a product of two of its subgroups H and K in the sense that $G = HK$. When speaking of G as an external product, group G is *isomorphic* to two of its subgroups H and K; that is, $G \cong H \rtimes K$ (in this case, $H \rtimes K$ consists of ordered pairs of elements of G whereas G consists of, well, elements of G!).

Note. Dummit and Foote's use of "recognition theorems" (Theorems DF.5.9 and DF.5.12) is reasonable since they let us *recognize* when a given group is isomorphic to a direct or semidirect product. However, they do not allow us to *construct* new groups from given groups. The direct product always produces a group (that is, the binary operation is always defined since it is just based on the binary operations of the groups in the product). The semidirect product allows us to produce a new

group from two given groups, provided we can find a homomorphism φ from K into $\mathrm{Aut}(H)$. So we might want to think of recognition theorems and internal products as a way to determine when *given groups* are isomorphic, whereas the operation of taking an external product allows us to produce *new groups*.

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