Supplement. A Proof of The Snake Lemma

Note. In his supplement, we give a proof of the Snake Lemma, stated in Section IV.1. Modules, Homomorphisms, and Exact Sequences; see Note IV.1.J. The proof given in this supplement is based on a document formerly posted online by Richard Blute of the University of Ottawa (accessed 10/20/2018). Unfortunately, Professor Blute has retired and this link does not currently (December 2023) work. He still maintains a personal website (accessed 12/17/2023), but the Snake Lemma document is apparently not posted there. We closely follow his presentation, but we largely use different symbols here. We present the proof in several steps; at each new step, we reuse symbols in new (but similar) roles.

The Snake Lemma. Let R be a ring and

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \Big\downarrow \alpha & & \Big\downarrow \beta & & \Big\downarrow \gamma & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

a commutative diagram of R-modules and R-module homomorphisms such that each row is an exact sequence. Then there is an exact sequence

$$\mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha) \to \mathrm{Coker}(\beta) \to \mathrm{Coker}(\gamma).$$

If, in addition, $f_A : A \to B$ is a monomorphism then so is the homomorphism $k_\alpha : A' \to B'$, and if $g_{B'} : B' \to C'$ is an epimorphism then so is $b_\beta : \mathrm{Coker}(\beta) \to \mathrm{Coker}(\gamma)$. Under these added conditions, we can extend the exact sequence on the left to include "$0 \to$" and on the right to include "$\to 0$."

Proof. First, let $f_A : A \to B$, $f_B : B \to C$, $g_{A'} : A' \to B'$, and $g_{B'} : B' \to C'$ be the R-module homomorphisms. Define $k_\alpha : \text{Ker}(\alpha) \to \text{Ker}(\beta)$ and $k_\beta : \text{Ker}(\beta) \to \text{Ker}(\gamma)$ as the restricted functions $f_A|_{\text{Ker}(\alpha)}$ and $f_B|_{\text{Ker}(\beta)}$, respectively. Notice that if $a \in \text{Ker}(\alpha)$ then

$$\begin{aligned} \beta(f_A(a)) &= g_{A'}(\alpha(a)) \text{ since the diagram is commutative} \\ &= g'_A(a)(0) = 0 \text{ since } g_{A'} \text{ is a homomorphism,} \end{aligned}$$

so in fact $k_\alpha = f_A|_{\text{Ker}(\alpha)}$ does map $\text{Ker}(\alpha)$ to $\text{Ker}(\beta)$ (and similarly $k_\beta = f_B|_{\text{Ker}(\beta)}$ maps as claimed). So the given diagram including the R-module homomorphisms is:

$$\begin{array}{ccccccc} & & A & \xrightarrow{f_A} & B & \xrightarrow{f_B} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{g_{A'}} & B' & \xrightarrow{g_{B'}} & C' & & \end{array}$$

Now, $\text{Coker}(\alpha) = A'/\text{Im}(\alpha)$ and $\text{Coker}(\beta) = B'/\text{Im}(\beta)$ by definition of "cokernel." Define $c_\alpha : \text{Coker}(\alpha) \to \text{Coker}(\beta)$ and $c_\beta : \text{Coker}(\beta) \to \text{Coker}(\gamma)$ as

$$c_\alpha(a' + \text{Im}(\alpha)) = g_{A'}(a') + \text{Im}(\beta) \text{ and } c_\beta(b' + \text{Im}(\beta)) = g_{B'}(b') + \text{Im}(\gamma).$$

We now show c_α and c_β are well-defined. If $a'_1 + \text{Im}(\alpha) = a'_2 + \text{Im}(\alpha)$, then $a'_1 - a'_2 \in \text{Im}(\alpha)$ so that $a'_1 - a'_2 = \alpha(a)$ for some $a \in A$. Then

$$\begin{aligned} g_{A'}(a'_1 - a'_2) &= g_{A'}(\alpha(a)) = (g_{A'} \circ \alpha)(a) \\ &= (\beta \circ f_A)(a) \text{ since the diagram is commutative} \\ &= \beta(f_A(a)) \in \text{Im}(\beta), \end{aligned}$$

so

$$c_\alpha(a'_2 + \text{Im}(\alpha)) = g_{A'}(a'_2) + \text{Im}(\beta) = g_{A'}(a'_2) + g_{A'}(a'_1 - a'_2) + \text{Im}(\beta)$$
$$= g_{A'}(a'_2) + g_{A'}(a'_1) - g_{A'}(a'_2) + \text{Im}(\beta) = g_{A'}(a'_1) + \text{Im}(\beta) = c_\alpha(a'_1 + \text{Im}(\alpha))$$

and so c_α is well-defined (that is, independent of the representative of the coset used). Similarly, c_β is well-defined. The sequence to be shown exact is then:

$$\text{Ker}(\alpha) \xrightarrow{k_\alpha} \text{Ker}(\beta) \xrightarrow{k_\beta} \text{Ker}(\gamma) \xrightarrow{\delta} \text{Coker}(\alpha) \xrightarrow{c_\alpha} \text{Coker}(\beta) \xrightarrow{c_\beta} \text{Coker}(\gamma).$$

We now define $\delta : \text{Ker}(\gamma) \to \text{Coker}(\alpha)$ (the "middle of the snake"). Let $c \in \text{Ker}(\gamma)$. Since the first row is exact, then f_B is an epimorphism ("onto"; because $f_C : C \to \{0\}$ implies $\text{Im}(f_B) =$

$\mathrm{Ker}(f_C) = C$), and so there is <u>some</u> $b \in B$ such that $f_B(b) = c$. Now $\beta(b) \in B'$. Since $c \in \mathrm{Ker}(\gamma)$ then

$$\begin{aligned} g_{B'}(\beta(b)) &= (g_{B'} \circ \beta)(b) \\ &= (\gamma \circ f_B)(b) \text{ since the diagram is commutative} \\ &= \gamma(f_B(b)) = \gamma(c) \text{ since } f_B(b) = c \\ &= 0 \text{ since } c \in \mathrm{Ker}(\gamma). \end{aligned}$$

So $\beta(b) \in \mathrm{Ker}(g_{B'}) = \mathrm{Im}(f_{A'})$ since the second row is exact. So $\beta(b) \in \mathrm{Ker}(g_{B'}) = \mathrm{Im}(f_{A'})$ since the second row is exact. So $\beta(b) = g_{A'}(a')$ for some $a' \in A'$. Introducing $g_0 : \{0\} \to A'$ (the inclusion map) and using the fact that the second row is exact, $\mathrm{Im}(g_0) = \mathrm{Ker}(g_{A'}) = \{0\}$ and so $g_{A'}$ is a monomorphism (one to one) by Theorem I.2.3 (see also Note IV.1.C) and so a' is the unique element of A' such that $\beta(b) = g_{A'}(a')$. Now define

$$\delta(c) = a' + \mathrm{Im}(\alpha).$$

Now we have determined a' as follows:

$$\begin{aligned} &f_B(b) = c \text{ for } \underline{\text{some}}\ b \in B \\ &\beta(b) = g_{A'}(a') \text{ for unique } a' \in A' \text{ (unique for given } b). \end{aligned} \qquad (*)$$

So we have $\delta(c) = (g_{A'})^{-1}(\beta(b)) + \mathrm{Im}(\alpha)$ where b is <u>some</u> element of the inverse image of $\{c\}$ under f_B, $b \in f_B^{-1}[\{c\}]$. So to show that δ is well-defined (this is the topic of discussion between Dr. Kate Gunzinger and Mr. Cooperman in the 1980 Rastar Films' *It's My Turn*) we need to consider the value of $\delta(c)$ for two different elements of $f_B^{-1}[\{c\}]$, say both b_1 and b_2 satisfy $f_B(b_1) = c$ and $f_B(b_2) = c$. As above, there are unique a_1' and a_2' such that $g_{A'}(a_1') = \beta(b_1)$ and $g_{A'}(a_1') = \beta(b_2)$. Notice that $f_B(b_1 - b_2) = f_B(b_1) - f_B(b_2) = c - c = 0$ so that $b_1 - b_2 \in \mathrm{Ker}(f_B)$. Since the first row is exact then $\mathrm{Im}(f_A) = \mathrm{Ker}(f_B)$, there is $a \in A$ such that $f_A(a) = b_1 - b_2$. Hence,

$$\begin{aligned} g_{A'}(\alpha(a)) &= (g_{A'} \circ \alpha)(a) \\ &= (\beta \circ f_A)(a) \text{ since the diagram is commutative} \\ &= \beta(f_A(a)) = \beta(b_1 - b_2) = \beta(b_1) - \beta(b_2) \\ &= g_{A'}(a_1') - g_{A'}(a_2') = g_{A'}(a_1' - a_2'). \end{aligned}$$

Since $g_{A'}$ is a monomorphism, then $a_1' - a_2' = \alpha(a) \in \mathrm{Im}(\alpha)$. Hence

$$a_2' + \mathrm{Im}(\alpha) = a_2' + \alpha(a) + \mathrm{Im}(\alpha) = a_2' + (a_1' - a_2') + \mathrm{Im}(\alpha) = a_1' + \mathrm{Im}(\alpha),$$

so δ is well-defined. We now show exactness and work left to right.

Since the top row is exact, then $\text{Im}(f_A) = \text{Ker}(f_B)$ and so $f_B \circ f_A$ is the zero function. Therefore $k_\beta \circ k_\alpha = f_B|_{\text{Ker}(\beta)} \circ f_A|_{\text{Ker}(\alpha)}$ is the zero function. So $\text{Im}(k_\alpha) \subset \text{Ker}(k_\beta)$. Conversely, suppose $b \in \text{Ker}(\beta)$ with $k_\beta(b) = 0$ (that is, $b \in \text{Ker}(k_\beta)$). Then $f_B(b) = 0$, so $b \in \text{Ker}(f_B) = \text{Im}(f_A)$ and hence $b = f_A(a)$ for some $a \in A$. We have

$$\begin{aligned} g_{A'}(\alpha(a)) &= (g_{A'} \circ \alpha)(a) \\ &= (\beta \circ f_A)(a) \text{ since the diagram is commutative} \\ &= \beta(f_A(a)) = \beta(b) = 0 \text{ since } b \in \text{Ker}(\beta). \end{aligned}$$

Since $g_{A'}$ is a monomorphism (one to one), then $\alpha(a) = 0$ and $a \in \text{Ker}(\alpha)$. So $k_\alpha(a) = f_A(a) = b \in \text{Im}(k_\alpha)$ and $\text{Ker}(k_\beta) \subset \text{Im}(k_\alpha)$. Hence $\text{Im}(k_\alpha) = \text{Ker}(k_\beta)$ and the sequence is exact at $\text{Ker}(\beta)$.

Let $b \in \text{Ker}(\beta)$ ($\text{Ker}(\beta)$ is the domain of k_β) so that $k_\beta(b)$ is an arbitrary element of $\text{Im}(k_\beta)$. Since $\beta(b) = 0$ then $\beta(b) = g_{A'}(0)$ (since $g_{A'} : A' \to B'$ is a homomorphism), so $\delta(k_\beta(b)) = 0 + \text{Im}(\alpha) = \text{Im}(\alpha) = 0 \in A'/\text{Im}(\alpha) = \text{Coker}(\alpha)$ (since $\beta(b) = g_{A'}(0)$ and $a' = 0$ in the notation of the definition of δ). Since $b \in \text{Ker}(\beta)$ is arbitrary (and $\text{Ker}(\beta)$ is the domain of $k(\beta)$) then $\delta(k_\beta(b)) = (\delta \circ k_\beta)(b) = 0$ for all $b \in \text{Ker}(\beta)$, and so $\text{Im}(k_\beta) \subset \ker(\delta)$. Conversely, suppose $c \in \text{Ker}(\gamma)$ ($\text{Ker}(\gamma)$ is the domain of δ) and $c \in \text{Ker}(\delta)$. Then $c = f_B(b)$ for some $b \in B$ (since the first row is exact and $\text{Im}(f_B) = c$; that is, f_B is an epimorphism because the kernel of the mapping $C \to \{0\}$ is all of C). By the definition of δ, since $c \in \text{Ker}(\delta)$, we have

$$\delta(c) = \text{Im}(\alpha) = 0 \in A'/\text{Im}(\alpha) = \text{Coker}(\alpha).$$

Since $c = f_B(b)$ then $\beta(b) = g_{A'}(a')$ for some $a' \in A'$ (see $(*)$ above), and since $\delta(c) = a' + \text{Im}(\alpha) = \text{Im}(\alpha)$, it must be that $a' \in \text{Im}(\alpha)$. Say $a' = \alpha(a)$ for $a \in A$. Then

$$\begin{aligned} \beta(f_A(a)) &= (\beta \circ f_A)(a) \\ &= (g_{A'} \circ \alpha)(a) \text{ since the diagram is commutative} \\ &= g_{A'}(\alpha(a)) = g_{A'}(a') \\ &= \beta(b) \text{ since } \beta(b) = g_{A'}(a). \end{aligned}$$

Since β is a homomorphism, $0 = \beta(b) - \beta(f_A(a)) = \beta(b - f_A(a))$ and $b - f_A(a) \in \text{Ker}(\beta)$. Finally,

$$\begin{aligned} k_\beta(b - f_A(a)) &= f_B(b - f_A(a)) \text{ since } b - f_A(a) \in \text{Ker}(\beta) \text{ and by the definition of } k_\beta \text{ as } f_B|_{\text{Ker}(\beta)} \\ &= f_B(b) - (f_B \circ f_A)(a) \text{ since } f_B \text{ is a homomorphism} \\ &= f_B(b) - 0 \text{ since } \text{Im}(f_A) = \text{Ker}(f_B) \text{ by the exactness of the first row} \\ &= f_B(b) = c \text{ since } c = f_B(b). \end{aligned}$$

That is, $c \in \text{Im}(k_\beta)$. Since c is an arbitrary element of $\text{Ker}(\delta)$ (in the domain $\text{Ker}(\gamma)$ of δ), then $\text{Ker}(\delta) \subset \text{Im}(k_\beta)$. Hence $\text{Im}(k_\beta) = \text{Ker}(\delta)$ and the sequence is exact at $\text{Ker}(\gamma)$.

Let $c \in \text{Ker}(\gamma)$ so that $\delta(c)$ is an arbitrary element of $\text{Im}(\gamma)$. Since f_B is an epimorphism by the exactness of the first row, then $c = f_B(b)$ for some $b \in B$. By $(*)$ above we have $\beta(b) = g_{A'}(a')$ for some $a' \in A'$. So by the definition of δ, $\delta(c) = a' + \text{Im}(\alpha)$. Then

$$\begin{aligned}
(c_\alpha \circ \delta)(c) &= c_\alpha(\delta(c)) = c_\alpha(a' + \text{Im}(\alpha)) \\
&= g_{A'}(a') + \text{Im}(\beta) \text{ by the definition of } c_\alpha \\
&= \beta(b) + \text{Im}(\beta) \text{ since } \beta(b) g_{A'}(a') \\
&= \text{Im}(\beta) \text{ since } \beta(b) \in \text{Im}(\beta) \\
&= 0 \in B'/\text{Im}(\beta).
\end{aligned}$$

Since c is an arbitrary element of $\text{Ker}(\gamma)$ (the domain of δ), then $c_\alpha \circ \delta$ is the zero function and $\text{Im}(\delta) \subset \text{Ker}(c_\alpha)$. Conversely, suppose $a' + \text{Im}(\alpha) \in A'/\text{Im}(\alpha) = \text{Coker}(\alpha)$ is in the kernel of c_α. Then

$$\begin{aligned}
c_\alpha(a' + \text{Im}(\alpha)) &= g_{A'}(a') + \text{Im}(\beta) \text{ by the definition of } c_\alpha \\
&= \text{Im}(\beta) = 0 \in A'/\text{Im}(\alpha) = \text{Coker}(\alpha) \text{ since } a' + \text{Im}(\alpha) \in \text{Ker}(c_\alpha),
\end{aligned}$$

and so $g_{A'}(a') \in \text{Im}(\beta)$, say $g_{A'}(a') = \beta(b)$ for some $b \in B$. Let $c = f_B(b)$. Then

$$\begin{aligned}
\gamma(c) &= \gamma(f_B(b)) = (\gamma \circ f_B)(b) \\
&= (g_{B'} \circ \beta)(b) \text{ since the diagram commutes} \\
&= g_{B'}(\beta(b)) = g_{B'}(g_{A'}(a')) = 0 \text{ since } \text{Im}(g_{A'}) = \text{Ker}(g_{B'}) \\
&\quad \text{by the exactness of the second row,}
\end{aligned}$$

so $c \in \text{Ker}(\gamma)$ (the domain of δ) and $\delta(c) = a' + \text{Im}(\alpha)$ by the definition of δ (and the choice of c). So $a' + \text{Im}(\alpha) \in \text{Im}(\delta)$. Since $a' + \text{Im}(\alpha)$ is an arbitrary element of $\text{Ker}(c_\alpha)$ then $\text{Ker}(c_\alpha) \subset \text{Im}(\delta)$. Hence $\text{Im}(\delta) = \text{Ker}(c_\alpha)$ and the sequence is exact at $\text{Coker}(\alpha)$.

Since the second row is exact, then $\text{Im}(g_{A'}) = \text{Ker}(g_{B'})$ and so $g_{B'} \circ g_{A'}$ is the zero function. Therefore for $a' + \text{Im}(\alpha) \in \text{Coker}(\alpha)$,

$$\begin{aligned}
(c_\beta \circ c_\alpha)(a' + \text{Im}(\alpha)) &= c_\beta(c_\alpha(a' + \text{Im}(\alpha))) \\
&= c_\beta(g_{A'}(a') + \text{Im}(\beta)) \text{ by the definition of } c_\alpha \\
&= g_{B'}(g_{A'}(a')) + \text{Im}(\gamma) \text{ by the definition of } c_\beta
\end{aligned}$$

$$\begin{aligned} &= (g_{B'} \circ g_A)(a') + \text{Im}(\gamma) = \text{Im}(\gamma) \text{ since } g_{B'} \circ g_{A'} \text{ is the zero function} \\ &= 0 \in C'/\text{Im}(\gamma) = \text{Coker}(\gamma). \end{aligned}$$

Since $a' + \text{Im}(\alpha)$ is an arbitrary element of $\text{Coker}(\alpha)$ (the domain of $c_\beta \circ c_\alpha$), then $c_\beta \circ c_\alpha$ is the zero function and $\text{Im}(c_\alpha) \subset \text{Ker}(c_\beta)$. Conversely, suppose $b' + \text{Im}(\beta) \in B'/\text{Im}(\beta) = \text{Cojer}(\beta)$ is in $\text{Ker}(c_\beta)$. Then

$$c_\beta(b' + \text{Im}(\beta)) = g_{B'}(b') + \text{Im}(\gamma) = \text{Im}(\gamma) = 0 \in B'/\text{Im}(\beta) = \text{Coker}(\beta),$$

$g_{B'}(b') \in \text{Im}(\gamma)$, and so $\gamma(c) = g_{B'}(b')$ for some $c \in C$. Since f_B is an epimorphism by the exactness of the first row, then $c = f_B(b)$ for some $b \in B$. Now $\beta(b) \in \text{Im}(\beta)$, so $b' + \text{Im}(\beta) = b' - \beta(b) + \text{Im}(\beta)$ in $B'/\text{Im}(\beta) = \text{Coker}(\beta)$. Now

$$\begin{aligned} g_{B'}(b' - \beta(b)) &= g_{B'}(b') - g_{B'}(\beta(b)) = g_{B'}(b') - (g_{B'} \circ \beta)(b) \\ &= g_{B'}(b') - (\gamma \circ f_B)(b) \text{ since the diagram is commutative} \\ &= \gamma(c) - \gamma(f_B(b)) \text{ since } \gamma(c) = g_{B'}(b') \\ &= \gamma(c) - \gamma(c) = 0 \text{ since } f_B(b) = c. \end{aligned}$$

We started with $b' + \text{Im}(\beta) = b' - \beta(b) + \text{Im}(\beta)$ as an arbitrary element of $\text{Coker}(\beta)$ and saw that $g_{B'}(b' - \beta(b)) = 0$, so without loss of generality we can assume $g_{B'}(b') = 0$ (just replace b' with $b' - \beta(b)$ as the representation of coset $b' + \text{Im}(\beta)$). That is, $b' \in \text{Ker}(g_{B'})$ without loss of generality. Since the second row is exact, then $\text{Im}(g_{A'}) = \text{Ker}(g_{B'})$ and so $b' \in \text{Im}(g_{A'})$. Hence $b' = g_{A'}(a')$ for some $a' \in A'$. Then

$$\begin{aligned} c_\alpha(a' + \text{Im}(\alpha)) &= g_{A'} + \text{Im}(\beta) \text{ by the definition of } c_\alpha \\ &= b' + \text{Im}(\beta), \end{aligned}$$

and so $b' + \text{Im}(\beta) \in \text{Im}(c_\alpha)$. Since $b' + \text{Im}(\beta)$ is an arbitrary element of $\text{Ker}(c_\beta)$, then $\text{Ker}(c_\beta) \subset \text{Im}(c_\alpha)$. Hence $\text{Im}(c_\alpha) = \text{Ker}(c_\beta)$ and the sequence is exact at $\text{Coker}(\beta)$. Therefore, the sequence

$$\text{Ker}(\alpha) \stackrel{k_\alpha}{\to} \text{Ker}(\beta) \stackrel{k_\beta}{\to} \text{Ker}(\gamma) \stackrel{\delta}{\longrightarrow} \text{Coker}(\alpha) \stackrel{c_\alpha}{\to} \text{Coker}(\beta) \stackrel{c_\beta}{\to} \text{Coker}(\gamma)$$

is exact.

Finally, if f_A is a monomorphism ("one to one"; in which case the first row of the diagram can be extended to the left to include "$0 \to$"), then $k_\alpha = f_A|_{\text{Ker}(\alpha)}$ is a monomorphism, as claimed. The exact sequence of kernels and cokernels can then be extended to the left to include "$0 \to$." If $g_{B'}$ is an epimorphism ("onto"; in which case the second row of the diagram can be extended to the right to include "$\to 0$") and $c' + \mathrm{Im}(\gamma) \in C'/\mathrm{Im}(\gamma) = \mathrm{Coker}(\gamma)$, then $g_{B'}(b') = c'$ for some $b' \in B'$. So

$$\begin{aligned} c_\beta(b' + \mathrm{Im}(\beta)) &= g_B(b') + \mathrm{Im}(\gamma) \text{ by the definition of } c_\beta \\ &= c' + \mathrm{Im}(\gamma), \end{aligned}$$

and $c' + \mathrm{Im}(\gamma) \in c_\beta$. Since $c' + \mathrm{Im}(\gamma)$ is an arbitrary element of $\mathrm{Coker}(\gamma)$, then $\mathrm{Im}(c_\beta) = \mathrm{Coker}(\gamma)$ and c_β is an epimorphism (onto), as claimed. The exact sequence of kernels and cokernels then can be extended to the right to include "$\to 0$." ■

Created: 10/20/2018; *Revised: 12/27/2023*