# Chapter II. The Structure of Groups
## Study Guide

The following is a brief list of topics covered in Chapter II of Hungerford's *Algebra*. This list is not meant to be comprehensive, but only gives a list of several important topics. You should also carefully study the proofs given in class and the homework problems.

**Section II.1. Free Abelian Groups.**
Linear combination, basis of an abelian group, free abelian group (Theorem II.1.1), cardinalities of bases of a free abelian group (Theorem II.1.2), rank of a free abelian group, the "Fundamental theorem of Free Abelian Groups" (Proposition II.1.3), abelian groups are homomorphic images of free abelian groups (Theorem II.1.4), subgroups of free abelian groups are free abelian (Theorem II.1.6).

**Section II.2. Finitely Generated Abelian Groups.**
Every finitely generated abelian group is isomorphic to a finite direct sum of cyclic groups (Theorems II.2.1 and II.2.3), the converse of Lagrange's Theorem holds for finite abelian groups (Corollary II.2.4), torsion group, torsion subgroup, torsion free, the Fundamental Theorem of Finitely Generated Abelian Groups (Theorem II.2.6), Betti number, invariant factors, isomorphic finitely generated abelian groups (Corollary II.2.7), finding invariant factors.

**Section II.3. The Krull Schmidt Theorem.**
Our notation of $H \times^i K$ for an internal direct product (Note II.3.A), complement of a subgroup in a group, indecomposable group, decomposable groups and normal subgroups (Note II.3.B), groups without proper normal subgroups are indecomposable so that simple groups are indecomposable (Note II.3.C), indecomposable groups which are not simple exist (Lemma II.3.A), ascending chain condition (ACC), descending chain condition (DCC), finiteness and ACC/DCC (Note II.3.E), ACC/DCC and direct products of a finite number of indecomposable subgroups (Theorem II.3.3), finite groups satisfying both ACC and DCC are a direct product of indecomposable subgroups (Note II.3.F), normal endomorphism, necessary and sufficient conditions in terms of ACC and DCC for an endomorphism to be an automorphism (Lemma II.3.4), Fitting's Lemma (Lemma II.3.5), nilpotent endomorphism, properties of endomorphisms in indecomposable groups that satisfy both ACC and DCC (Corollary II.3.6 and Note II.3.G), sums of normal nilpotent epimorphisms of an indecomposable group that satisfies ACC and DCC is nilpotent (Corollary II.3.7), canonical epimorphism

associated with an internal direct product of a group, the Krull-Schmidt Theorem (Theorem II.3.8), every finite group is isomorphic to a unique direct product of a finite number of indecomposable subgroups (Note II.3.I), history of the Krull-Schmidt Theorem (Note II.3.J).

**Section II.4. The Action of a Group on a Set.**

Definition of action of a group on a set, left translation, conjugation, group action determines an equivalence relation (Theorem II.4.2), orbit of an element under group action $\overline{x}$, stabilizer $G_x$, conjugacy class, centralizers and normalizers ($C_H(x)$, $C_G(x)$, $N_H(K)$, $N_G(K)$), $|\overline{x}| = [G : G_x]$ (Theorem II.4.3), the class equation, Cayley's Theorem (every group is a group of permutations, Corollary II.4.6), inner automorphism, center of $G$, normal subgroup of index $p$ (Corollary II.4.10).

**Section II.5. The Sylow Theorems.**

Peter Luvig Sylow (1832–1918), $|S| \equiv |S_0| \pmod{p}$ (Lemma II.5.1), Cauchy's Theorem (Theorem II.5.2), $p$-group and $p$-subgroup, classification of $p$-groups (Corollary II.5.3), the First Sylow Theorem (Theorem II.5.7), Sylow $p$-subgroup, properties of Sylow $p$-subgroups (Corollary II.5.8), the Second Sylow theorem (Theorem II.5.9), the Third Sylow Theorem (Theorem II.5.10), use of Sylow Theorems to show that a group is not simple.

**Section II.6. Classification of Finite Groups.**

Groups of order $pq$ (Proposition II.6.1 and Corollary II.6.2), metacyclic group, groups of order 8 (Proposition II.6.3), groups of order 12 (Proposition II.6.4), dicyclic groups $\mathrm{Dic}_n$, groups of order up to 15, groups of order $p^2$, groups of order $p^3$ (Heisenberg group).

**Direct Products and Semidirect Products.**

Commutator $[x, y]$, commutator subgroup, characteristic subgroup $G'$, the Recognition Theorem for Direct Products (Theorem DF.5.9), internal direct product, external direct product, when $HK$ is a subgroup of $G$ (Corollary DF.3.15), semidirect product of $H$ and $K$ with respect to homomorphism $\varphi$ (Theorem DF.5.10), the dicyclic group $\mathrm{Dic}_3$ as $\mathbb{Z}_3 \rtimes_\varphi \mathbb{Z}_4$, nonabelian group of order $pq$ interpreted as $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_q$, the Recognition Theorem for Semidirect Products, groups of order 30.

**Section II.7. Nilpotent and Solvable Groups.**

The group $C_i(G)$, ascending central series, nilpotent group, $p$-groups are niplotent (Proposition II.7.2), finite products of nilpotent groups are nilpotent (Proposition II.7.3), classification of finite milpotent groups (Proposition II.7.5), the converse of Lagrange's Theorem for finite nilpotent groups (Corollary II.7.6), commutator subgroup $G'$, $G/G'$ is abelian (Theorem II.7.8), the $i$th derived subgroup $G^{(i)}$, solvable group (Definition II.7.9), nilpotent groups are solvable (Proposition II.7.10), some properties of solvable groups (Theorem II.7.11), $S_n$ not solvable for $n \geq 5$ (Corollary II.7.12), characteristic subgroup, fully invariant subgroup, minimal normal subgroup, relations between subgroups (Lemma II.7.13), the "Sylow-like" theorem for solvable groups (Proposition II.7.14).

**Section II.8. Normal and Subnormal Series.**
Subnormal series, factors, normal series, refinement of a series, composition series, solvable series, maximal normal subgroup, properties of series (Theorem II.8.4), Fraleigh and Hungerford have equivalent definitions of solvable groups (Theorem II.8.5), classification of finite solvable groups (Proposition II.8.6), equivalent subnormal series, Zassenhaus Lemma/Butterfly Lemma (Lemma II.8.9), Schrier's Theorem (Theorem II.8.10), Jordan-Hölder Theorem (Theorem II.8.11), the importance of finite simple groups.

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