Section V.1.Appendix.

Ruler and Compass Constructions

Note. In this section, we explore "straight edge and compass" constructions. Hungerford's expression "ruler" is misleading in that we don't actually measure lengths using a ruler.

Note. All constructions are performed in the 2-dimensional plane. We start with a line segment defined to be of unit length. For reference, we assume the presence of x and y axes. We then adhere to the following rules of construction:

- We then can construct other lines or line segments using a straight edge through two constructed points.
- Given a point p and a line segment of a given length l, we can use the compass to construct a circle with center p and radius l.
- Given a line segment of a certain length, a line segment of the same length can be constructed on any given line.

A point is *constructed* when it results from the intersection of two lines, two circles, or a line and a circle.

Note. In the past, a study of straight edge and compass constructions were a standard part of a high school geometry class. Since this may not to be the case universally these days, I have prepared a YouTube video "Compass and Straight Edge Constructions" to explain the basic ideas. See:

`https://www.youtube.com/watch?v=S24GYj1rWGs`

(accessed 2/26/2015). This video supplement is based on Section 32, "Geometric Constructions," of Fraleigh's *A First Course in Abstract Algebra*, 7th Edition.

Note. Euclid's *Element's of Geometry* depends, philosophically at least, very heavily on the idea of compass and straight edge constructions. In fact, the very first result, Proposition 1 of Book I, is a demonstration of the construction of an equilateral triangle using a compass and straight edge. The proposition states: "On a given finite straight line, to construct an equilateral triangle." This is typical wording for a result in the *Elements*, and reflects the idea that a geometric object does not exist unless it can be constructed. In fact, Richard Dedekind (1831–1916) is credited with claiming that continuity of the plane (more appropriately, "completeness" in the sense explained in the notes for the appendix to Section V.3) is not essential for the constructions of Euclid and he defines a discontinuous space in which the constructions can be made. Wendell Strong explores this in "Is Continuity of Space Necessary to Euclid's Geometry?" in *The Bulletin of the American Mathematical Society*, **4**(9), 443–448 (1898). He calls the space "quadratic space." It turns out that quadratic space consists of all points (x, y) in the Cartesian plane where x and y are constructible numbers—a concept to be elaborated on in this appendix (in Proposition V.1.16). Strong states that an "essential feature of the quadratic space is that it is the least space [in the set theoretic inclusion sense] in which the constructions of Euclid are possible."

Note. As shown in the YouTube video "Compass and Straight Edge Constructions," some of the constructions we can perform include:

1. Construction of an equilateral triangle.

2. Bisection of a line segment.

3. Bisection of an angle.

4. Construct a perpendicular to a line through a given point not on the line.

5. Given a line and a point not on the line, a parallel to the line passing through the point can be constructed.

6. A line segment can be trisected (or, in general, cut into any number of equal length pieces).

Note. There are three famous construction problems that *cannot* be performed:

(a) Trisection of an arbitrary angle.

(b) Doubling the cube (that is, construction of a cube which is twice the volume of a given cube).

(c) Squaring the circle (that is, given a circle to construct a square with the same area)—Hungerford does not mention this problem).

It was Pierre Wantzel in 1837 who first showed that trisecting an angle and doubling the cube are impossible in "Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas" in *Journal de*

Mathématiques Pures et Appliquées **1**(2), 366-372. In 1882, Ferdinand Lindemann proved that π is transcendental in "*Über die Zahl* π," *Mathematische Annalen* **20**, 213–225 (1882), from which the impossibility of squaring the circle follows. See the historical note on page 298 of Fraleigh's 7th Edition. We now use field theory to demonstrate these claims.

Definition. Let F be a subfield of $\mathbb{R}$. The *plane of* F is $F \times F \subset \mathbb{R} \times \mathbb{R}$. If P and Q are distinct points in the plane F, then the unique line through P and Q is a *line in* F and the circle with center P and radius the line segment PQ is a *circle in* F.

Note. We have, by Exercise V.1.24, that every line in F, say $L = \{(x,y)\}$, has an equation of the form $ax + by + c = 0$ where $a, b, c \in F$ and every circle in F, say $C = \{(x,y)\}$, has an equation of the form $x^2 + y^2 + ax + by + c = 0$ where $a, b, c \in F$.

Lemma V.1.15. Let F be a subfield of the field $\mathbb{R}$ of real numbers and let L_1, L_2 be nonparallel lines in F and C_1, C_2 distinct circles in F. Then

(i) $L_1 \cap L_2$ is a point in the plane of F;

(ii) $L_1 \cap C_1 = \varnothing$ or consists of one or two points in the plane of $F(\sqrt{u})$ for some $u \in F$ where $u \geq 0$;

(iii) $C_1 \cap C_2 = \varnothing$ or consists of one or two points in the plane of $F(\sqrt{u})$ for some $u \in F$ where $u \geq 0$.

Definition. A real number c is *constructible* if the point $(c, 0)$ can be constructed by a finite sequence of straight edge and compass constructions that begin with points with integer coordinates.

Note. The constructibility of c is equivalent to the construction of a segment of length $|c|$. The following hold:

(i) Every rational number is constructible.

(ii) If $c \geq 0$ is constructible, then so is $\sqrt{c}$.

(iii) If c and d are constructible, then $c \pm d$, cd, and c/d (where $d \neq 0$) are constructible. Therefore the constructible numbers form a field.

These claims are justified in Exercise V.1.25. See also the YouTube video "Compass and Straight Edge Constructions;" For (i) and (iii) see Theorem 32.1 and for (ii) see Theorem 32.6′.

Proposition V.1.16. If a real number c is constructible, then c is algebraic of degree a power of 2 over the field $\mathbb{Q}$ or rationals.

Corollary V.1.17. Straight Edge and Compass Trisection of a General Angle is Impossible.

An angle of $60°$ cannot be trisected by ruler and compass constructions, and therefore a general angle cannot be trisected.

Corollary V.1.18. Straight Edge and Compass Doubling of the Cube is Impossible.
It is impossible by ruler and compass constructions to duplicate a cube of side length 1 (that is, to construct the side of a cube of volume 2).

Corollary V.1.19. Straight Edge and Compass Squaring of the Circle is Impossible.
It is impossible by ruler and compass constructions to construct a square with area equal to the area of a circle of radius 1 (that is, to construct a square with area π).

Note. The process developed in this section can also be used to give necessary and sufficient conditions for the construction of a regular n-gon. Surprisingly, this involved the first straight edge and compass construction of a 17-gon by Gauss (after this problem had set unsolved for the two millenia since the publication of the *Elements*). For more details, see:

`http://faculty.etsu.edu/gardnerr/4127/notes/VI-32.pdf`

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