Part VII. Linear Algebra

Note. In this chapter, we present several of the topics covered in sophomore Linear Algebra (MATH 2010), but in a much more general setting. The role of finite dimensional vector spaces are replaced with free modules and matrices (which represent linear transformations in linear algebra) are related (isomorphic) to homomorphisms between free modules.

Section VII.1. Matrices and Maps

Note. In this section, we define matrices with entries in a ring and show that all matrices of the same size form an R-R bimodule (in Theorem 1.1), and that a homomorphism between free (left) modules is isomorphic to matrix mapping one module to the other (in Theorem 1.2). We define equivalent matrices and relate them to homomorphism of free (left) modules in terms of different bases of the modules (see Theorem 1.6).

Definition. Let R be a ring. An array of elements of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{pmatrix},$$

with $a_{ij} \in R$, n rows, and m columns, is a $n \times m$ *matrix* over R. An $n \times n$ matrix is a *square matrix*. We denote matrix A as (a_{ij}) where a_{ij} is the entry in the ith row and jth column of A. Two $n \times m$ matrices (a_{ij}) and (b_{ij}) are *equal* if $a_{ij} = b_{ij}$ in R for all $1 \leq i \leq n$ and $1 \leq j \leq m$. The elements $a_{11}, a_{22}, a_{33}, \ldots$ form the *main diagonal* of (a_{ij}). An $n \times n$ matrix with $a_{ij} = 0$ for $i \neq j$ is a *diagonal matrix.* If R has an identity element 1_R, then the *identity matrix* I_n is the $n \times n$ diagonal matrix with 1_R in each entry on the main diagonal; that is, $I_n = (\delta_{ij})$ where δ is the Kronecker delta. The $n \times m$ matrices with all entries 0 are the *zero matrices.* The set of all $n \times n$ matrices over R is denoted $\mathrm{Mat}_n(R)$. The *transpose* of an $n \times m$ matrix $A = (a_{ij})$ is the $m \times n$ matrix $A^t = b)ij$ such that $b_{ij} = a_{ji}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Note. We next define the familiar algebraic structure on the set of all $n \times m$ matrices.

Definition. If $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times m$ matrices, then the *sum* $A + B$ is defined to be the $n \times m$ matrix $C = (c_{ij})$, where $c_{ij} = a_{ij} + b_{ij}$. If $A = (a_ij)$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix then the *product* AB is the $m \times p$ matrix $C = (c_{ij})$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. If $A = (a_{ij})$ is an $n \times m$ matrix and $r \in R$ then rA is the $n \times m$ matrix (ra_{ij}) and Ar is the $n \times m$ matrix $(a_{ij}r)$. The matrix rI_n is a *scalar matrix.*

Note. The previous two definitions contain nothing that you don't see in sophomore Linear Algebra (MATH 2010), except for the need to deal with scalar multiplication of a matrix on the left and the right (due to the fact that we do not assumer commutativity of multiplication). If ring R is commutative, then products and transposes interact in the usual way, $(AB)^t = B^t A^t$, but this may not be the case if R is noncommutative, as is to be shown in Exercise VII.1.1.

Note. Recall from Section IV.4. Hom and Duality that for rings R and S, an abelian group A is an R-S *bimodule* if A is both a left R-module, a right S-module, and $r(as) = (ra)s$ for all $a \in A, r \in R, s \in S$. We leave the proof of the follwing as Exercise VII.1.A.

Theorem VII.1.1. If R is a ring, then the set of all $n \times m$ matrices over R forms an R-R bimodule under addition, with the $n \times m$ zero matrix as the additive identity. Multiplication of matrices, when defined, is associative and distributive over addition. For each $n > 0$, $\text{Mat}_n(R)$ is a ring. If R has an identity, so does $\text{MAT}_n(R)$ (namely, the identity matrix I_n).

Note. For the remainder of this section, we assume that all rings have an identity. We will need an identity matrix and hence an identity in R.

Note. Before proving the next theorem, it might be a good idea to review the ideas of an ordered basis and a coordinate vector relative to an ordered basis. See my online notes for Linear Algebra (MATH 2010) on Section 3.3. Coordinatization of Vectors. These ideas are essential for proving the Fundamental Theorem of Finite Dimensional Vector Spaces (see Theorem 3.3.A in the Linear Algebra notes).

Theorem VII.1.2. Let R be a ring with identity. Let E be a free left R-module with a finite basis of n elements and F a free left R-module with a finite basis of m elements. Let M be the left R-module of all $n \times m$ matrices over R. Then there is an isomorphism of abelian groups establishing that $\text{Hom}_R(E, F) \cong M$. If R is commutative this is an isomorphism of left R-modules.

Definition. Let R be a ring with identity, E a free left R-module with basis $U = \{u_1, u_2, \ldots, u_n\}$, F a free left R-module with finite basis $V = \{v_1, v_2, \ldots, v_m\}$, M be the left R-module of all $n \times m$ matrices over R, and additive homomorphism $\beta : \text{Hom}_R(E, F) \to M$ mapping $f \mapsto A$ where A is the $n \times m$ matrix (r_{ij}) where the r_{ij} are the coefficients in the system of equations in the proof of Theorem VII.1.2 (see also below). The $n \times m$ matrix $(r_{ij}) = \beta(f)$ is the *matrix of homomorphism* $f \in \text{Hom}_R(E, F)$ *relative to the ordered bases* U *of* E *and* V *of* F. When $E = F$ and $U = V$, $(r_{ij}) = \beta(f)$ is the *matrix of the endomorphism* f *relative to the ordered basis* U.

Note. The system of equations in the proof of Theorem VII.1.2 is

$$\begin{aligned}
f(u_1) &= r_{11}v_1 + r_{12}v_2 + \cdots + r_{1m}v_m \\
f(u_2) &= r_{21}v_1 + r_{22}v_2 + \cdots + r_{2m}v_m \\
&\vdots \\
f(u_n) &= r_{n1}v_1 + r_{n2}v_2 + \cdots + r_{nm}v_m.
\end{aligned}$$

So the ith row of $r_{ij} = \beta(f)$ consists of the coefficients of $f(u_i) \in F$ relative to the ordered basis $\{v_1, v_2, v_3, \ldots, v_m\}$

Note.

Revised: 11/2/2022