Chapter VIII. Commutative Rings and Modules

Section VIII.1. Chain Conditions

Note. This chapter serves as an introduction to commutative algebra. In this first section, however, we consider rings that may not be commutative and may not have identity. This section is necessary for a study of arbitrary rings, as given in Chapter IX, "The Structure of Rings."

Note. Recall that, informally, a module is like a vector space where the scalars come from a ring and the vectors come from an abelian group (see Definition IV.1.1).

Definition VIII.1.1. A module A satisfies the *ascending chain condition (ACC) on submodules* if for every chain $A_1 \subset A_2 \subset A_3 \subset \cdots$ of submodules of A, there is $n \in \mathbb{N}$ such that $A_i = A_n$ for all $i \geq n$. Such a module is called *Noetherian*. A module B satisfies the *descending chain condition (DCC) on submodules* if for every chain $B_1 \supset B_2 \supset B_3 \supset \cdots$ of submodules o fB, there is $m \in \mathbb{N}$ such that $B_i = B_m$ for all $i \geq m$. Such a module is also called *Artinian*.

Note. The previous definitions of ACC and DCC for modules is very similar to the definitions of ACC and DCC, respectively, for groups (though for groups, the chains are of *normal* subgroups, not simply groups); see Definition II.3.2.

Example. Notice that the additive group $\mathbb{Z}$ is a $\mathbb{Z}$-module (where the scalars are form ring $\mathbb{Z}$ and the vectors are from additive group $\mathbb{Z}$). As shown in Exercise II.3.5, $\mathbb{Z}$ satisfies the ACC but not the DCC. That is, $\mathbb{Z}$ is Noetherian but not Artinian.

Example. Recall the additive abelian group $Z(p^\infty) = \mathbb{Q}/\mathbb{Z}$, the *group of rationals modulo one* or the *Prüfer group*, from Section I.1. This is also a $\mathbb{Z}$-module (where the scalars are from ring $\mathbb{Z}$ and the vectors are from additive abelain group $Z(p^\infty)$). As shown in Exercise II.3.13, $Z(p^\infty)$ satisfies the DCC but not he ACC That is, $Z(p^\infty)$ is Artinian but not Noetherian.

Note. We now shift from chain conditions on modules to chain conditions on rings. To do so, we consider ring R as a left (or right) module over itself. So both the vectors and scalars come from R. Then if R' is a submodule of module R then R' must be an R-module itself. That is, for any scalar $r \in R$ and any vector $s \in R'$ we must have $rs \in R'$ in the case of R as a left R-module, or $sr \in R'$ in the case of R as a right R-module. So with this interpretation, we see that the submodules are in fact ideals of R (see Definition III.2.1 for left ideal and right ideal). Recall from Section III.2 that ideals are to rings as normal subgroups are to groups; see Theorem III.2.7 for motivation of this comment. So given this observation and the definitions of ACC and DCC for groups (Definitions II.3.2), the following definition is well-motivated.

Definition VIII.1.2. A ring R is *left* (respectively, *right*) *Noetherian* if R satisfies the ascending chain condition on left (respectively, right) ideals. R is *Noetherian* if R is both left and right Noetherian. A ring is *left* (respectively, *right*) *Artinian* if R satisfies the descending chain condition on left (respectively, right) ideals. R is *Artinian* if R is both left and right Artinian.

Note. Most of the results related to ACC and DCC in this section are stated in terms of modules, but by the previous definition these results also apply to rings.

Example. By Exercise III.2.7, ring R with identity is a division ring if and only if R has no proper left (or right) ideals. So the only ascending chain for division ring D is $\{0\} \subset D$ and the only descending chain is $D \supset \{0\}$. So a division ring is both Noetherian and Artinian.

Example. By Lemma III.3.6, if R is a principal ideal ring and $(a_1) \subset (a_2) \subset (a_3) \subset \cdots$ is a chain of (principal) ideals in R, then for $n \in \mathbb{N}$, $(a_j) = (a_n)$ for all $j \geq n$. If R is a commutative ring then each left ideal is also a right ideal (or simply an ideal) and so for commutative principal ideal ring R we see that R satisfies the ACC and so is Noetherian. Now $\mathbb{Z}$ and $\mathbb{Z}_m$ are commutative principal ideal rings (by Exercises III.2.20(1) and (c)), so $\mathbb{Z}$ and $\mathbb{Z}_m$ are Noetherian. By Exercise III.5.A, if F is a field then $F[x]$ is a principal ideal domain and so $F[x]$ is Noetherian.

Example. We'll see in Corollary VIII.1.12 that if D is a division ring, then the ring $\mathrm{Mat}_n(D)$ of all $n \times n$ matrices over D is both Artinian and Noetherian.

Note. In Exercise VIII.1.1(a), it is to be shown that a subring of $\mathrm{mat}_2(\mathbb{Q})$ is right Noetherian but not left Noetherian. In Exercise VIII.1.1(b), it is to be shown that a subring of $\mathrm{Mat}_2(\mathbb{R})$ is right Artinian but not left Artinian. We mentioned above (based on Exercise II.3.5) that $\mathbb{Z}$ is Noetherian but not Artinian. In Exercise IX.3.13 it is to be shown that every left (respectively, right) Artinian ring with identity is left (respectively right) Noetherian.

Note. Recall from Section 0.7, "The Axiom of Choice, Order, and Zorn's Lemma," that if $\leq$ is a partial ordering on set A (so that $\leq$ is reflexive, transitive, and antisymmetric) then $a, b \in A$ are *comparable* if either $a \leq b$ or $b \leq a$.

Definition. Let $(A, \leq)$ be a partially ordered set. An element $a \in A$ is *maximal* in A if for every $c \in A$ which is comparable to a, we have $c \leq a$. That is, $a \in A$ is maximal in A if $a \leq c$ for $c \in A$ implies $a = c$. An element $b \in A$ is *minimal* in A if for every $c \in A$ which is comparable to a we have $b \leq c$. That is, $b \in A$ is minimal in A if $c \leq b$ for $c \in A$ implies $c = b$.

Note. In this section we use the partial ordering of subset inclusion and put this on submodules of a given module (or ideals of a given ring).

Definition VIII.1.3. A module A satisfies the *maximum condition* (respectively, *minimum condition*) *on submodules* if every nonempty set of submodules of A contains a maximal (respectively, minimal) element with respect to set theoretic inclusion (i.e., subset inclusion).

Note. We now prove that a module satisfies the ACC/DCC if and only if it satisfies the maximum/minimum condition on submodules. Notice that the proof uses the Axiom of Choice.

Theorem VIII.1.4. A module A satisfies the ascending (respectively, descending) chain condition on submodules if and only if A satisfies the maximal (respectively, minimal) condition on submodules.

Note. Recall the definition of an exact sequence:

> **Definition IV.1.16.** A pair of module homomorphisms, $A \xrightarrow{f} B \xrightarrow{g} C$, is *exact* at B provided $\mathrm{Im}(f) = \mathrm{Ker}(g)$. A finite sequence of module homomorphisms, $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n$, is *exact* provided $\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$ for $i = 1, 2, \ldots, n-1$.

An exact sequence of the form $\{0\} \to A \xrightarrow{f} B \xrightarrow{g} \{0\}$ is a *short exact sequence*. Notice that f is a monomorphism (one to one) and g is an epimorphism (onto). We will use The Short Five Lemma, Lemma IV.1.17, in the proof of the following.

Theorem VIII.1.5. Let $\{0\} \to A \xrightarrow{f} B \xrightarrow{g} C \to \{0\}$ be a short exact sequence of modules. Then B satisfies the ascending (respectively, descending) chain condition on submodules if and only if A and C satisfy it.

Note. If A is a submodule of B and B satisfies the ascending (respectively, descending) chain condition then A satisfies it since each ascending (respectively, descending) chain of submodules of A is an ascending (respectively, descending) chain of B. The next result gives conditions under which the ACC or DCC on submodule A implies that B also has this condition.

Corollary VIII.1.6. If A is a submodule of a module B, then B satisfies the ascending (respectively, descending) chain condition if and only if A and B/A satisfy it.

Corollary VIII.1.7. If $A_1, A_2, \ldots, A_n$ are modules, then the direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ satisfies the ascending (respectively, descending chain condition on submodules if and only if each A_i satisfies it.

Note. The next theorem relates the "Noetherian-ness/Artinian-ness" of a ring with identity R to the ACC/DCC-ness of associated (finitely generated) R-modules. Its proof requires results from Section IV.2, "Free Modules and Vector Spaces." We start by recalling these results and some module-related definitions.

Note. Recall that a *unitary* R-module is an R-module where $1_R \in R$ and $1_R a = a$ for all "vectors" in group A. Also:

> **Definition IV.1.4.** If X is a subset of a module A over a ring R, then the intersection of all submodules of A containing X is the *submodule generated by* X (or "spanned" by X).

If set X in this definition is finite then the module generated by X is *finitely generated*. We have:

> **Corollary IV.2.2.** Every unitary module A over a ring R with identity is the homomorphic image of a free R-module F. If A is finitely generated then F may be chosen to be finitely generated.

A *free R-module* is one satisfying any one of the equivalent conditions of Theorem IV.2.1 (part (iii)) is that a free module F is R-module isomorphic to a direct sum of copies of the left R-module R. Though Theorem IV.2.1 doesn't explicitly mention finitely generated R-modules, we see in the proof of Theorem IV.2.1 (the (iv) implies (iii) part) that if X is a basis of R-module F then F is isomorphic to the direct sum $\sum R$ where there is one copy of R for each $x \in x$. So if F is finitely generated then it has a finite basis and is isomorphic to a direct sum of a *finite* number of copies of R.

Theorem VIII.1.8. If R is a left Noetherian (respectively, Artinian) ring with identity, then every finitely generated unitary left R-module A satisfies the ascending (respectively, descending) chain condition on the submodules. This also holds if "left" is replaced with "right."

Note. So far, we have seen that each result involving the ACC has an analogous result involving the DCC. the next theorem for a module satisfying the ACC on on submodules which "has no analogue for the descending chain condition" (see Hungerford, page 374).

Theorem VIII.1.9. A module A satisfies the ascending chain condition on submodules if and only if every submodule of A is finitely generated. In particular, a commutative ring R is Noetherian if and only if every ideal of R is finitely generated.

Note. We now carry over the ideas of subnormal series for groups of Section II.8, "Normal and Subnormal Series," to modules. The objective here is to prove that $\text{Mat}_n(D)$, wher D is a division ring, is both Artinian and Noetherian (in Corollary VIII.1.12). This result will be used in Chapter IX, "The Structure of Rings." We give definitions and state results without proof, since the proofs parallel the corresponding results for groups.

Definition. A *normal series* for a module A is a chain of submodules of the form $A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n$. The *factors* of the series are the quotient modules A_i/A_{i+1} for $i = 0, 1, 2, \ldots, n-1$. The *length* of the series is the number of proper inclusions, which equals the number of nontrivial factors. A *proper refinement* is one which has length longer than the original series. Two normal series are *equivalent* if there is a one-to-one correspondence between the nontrivial factors such that the corresponding <u>factors</u> are are isomorphic modules. A *composition series* for A is a normal series $A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n = \{0\}$ such that each factor A_k/A_{k+1}, for $k = 0, 1, 2, \ldots, n-1$, is a nonzero module with no proper submodules. If R has identity then a nonzero unitary module with no proper submodules is *simple*.

Note. If ring R has identity then a composition series is a normal series $A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n = \{0\}$ with simple factors.

Note. We claim without proof:

> **Theorem VIII.1.A.** If A is a module with a composition series, then there is no proper refinement of the composition series and therefore is equivalent to any of its refinements.

This is an analogue of Lemma II.8.8.

> **Theorem VIII.1.10.** Let A be a module.
>
> **(a)** Any two normal series of A have refinements that are equivalent.
>
> **(b)** Any two composition series of A are equivalent.

Note. Theorem VIII.1.10(a) is analogous to Schrier's Theorem (Theorem II.8.10) and Theorem VIII.1.10(b) is analogous to the Jordan-Hölder Theorem (Theorem II.8.11).

Note. We saw in Theorem II.8.4(i) that every finite group has a composition series. We now give necessary and sufficient conditions for a nonzero module to have a composition series (notice the use of the Axiom of Choice in the proof). We'll use this result to consider $\mathrm{Mat}_n(D)$ where D is a division ring.

Theorem VIII.1.11. A nonzero module A has a composition series if and only if A satisfies both the ascending and descending chain conditions on submodules.

Note. Now for the result we will use in our study of rings in Chapter IX. The proof requires some information from Chapter VII, "Linear Algebra," which we state within the proof as needed.

Corollary VIII.1.12. If D is a division ring, then the ring $\mathrm{Mat}_n(D)$ of all $n \times n$ matrices over D is both Artinian and Noetherian.

Revised: 8/27/2018