Chapter III. Elementary Properties and Examples of Analytic Functions

III.1. Power Series—Proofs of Theorems
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Proposition III.1.1

Proposition III.1.1. If $\sum a_n$ converges absolutely, then the series converges.

Proof. Let $\varepsilon > 0$. Let $z_n$ be the partial sum of $\sum_{k=1}^{\infty} a_k$:

$$z_n = a_1 + a_2 + \cdots + a_n.$$ 

Since $\sum_{n=1}^{\infty} |a_n|$ converges by hypothesis, then there is $N \in \mathbb{N}$ such that

$$\left| \sum_{k=1}^{N-1} |a_k| - \sum_{n=1}^{\infty} |a_n| \right| = \sum_{n=N}^{\infty} |a_n| < \varepsilon.$$ 


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So if $m > k \geq N$ then by the Triangle Inequality,

$$|z_m - z_k| = \left| \sum_{n=k+1}^{m} a_n \right| \leq \sum_{n=k+1}^{m} |a_n| \leq \sum_{n=N}^{\infty} |a_n| < \varepsilon,$$

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and so $\{z_n\}$ is a Cauchy sequence of complex numbers. Since $\mathbb{C}$ is complete by Proposition II.3.6, then $z_n \to z$ for some $z \in \mathbb{C}$. That is, there is $z \in \mathbb{C}$ with $\sum_{n=1}^{\infty} a_n = z$. 

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\[
\frac{1}{R} = \lim_{n \to \infty} |a_n|^{1/n} \quad \text{(so } 0 \leq R \leq \infty\text{).}
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Then
(a) if \( |z - a| < R \), the series converges absolutely,
(b) if \( |z - a| > R \), the series diverges, and
(c) if \( 0 < r < R \) then the series converges uniformly on \( |z - a| \leq r \). Moreover, \( R \) is the only number having properties (a) and (b). \( R \) is called the radius of convergence of the power series.

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Proof. Without loss of generality, \( a = 0 \).

(a) If \( |z| < R \), there is \( r \) with \( |z| < r < R \). Then \( 1/R < 1/r \) and so there
exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( |a_n|^{1/n} < 1/r \) (by definition of

\( \lim |a_n|^{1/n} \)).
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\[
\sum_{n=0}^{\infty} |a_nz^n| = \sum_{n=0}^{N-1} |a_nz^n| + \sum_{n=N}^{\infty} |a_nz^n| < \sum_{n=0}^{N-1} |a_nz^n| + \sum_{n=N}^{\infty} \left( \frac{|z|}{r} \right)^n.
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Theorem III.1.3 (continued 1)

**Theorem III.1.3.** If \( \sum_{n=0}^{\infty} a_n(z - a)^n \), define the number \( R \) as

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**Proof (continued).** Next,

\[
\sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{N-1} |a_n z^n| + \sum_{n=N}^{\infty} |a_n z^n| < \sum_{n=0}^{N-1} |a_n z^n| + \sum_{n=N}^{\infty} \left( \frac{|z|}{r} \right)^n.
\]

Since \( |z|/r < 1 \), then \( \sum_{n=N}^{\infty} a_n z^n \) converges absolutely, and the power series converges absolutely for \( |z| < R \).
Theorem III.1.3 (continued 2)

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(c) if \( 0 < r < R \) then the series converges uniformly on \( |z - a| \leq r \).

Proof (continued). (c) Suppose \( r < R \) and choose \( \rho \) such that \( r < \rho < R \). As in the proof of (a), let \( N \in \mathbb{N} \) be such that \( |a_n| < 1/\rho^n \) for all \( n \geq N \). Then if \( |z| \leq r \), we have \( |a_n z^n| < (r/\rho)^n \) for all \( n \geq N \), and \( (r/\rho) < 1 \).
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Theorem III.1.3. If \( \sum_{n=0}^{\infty} a_n(z - a)^n \), define the number \( R \) as

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\frac{1}{R} = \lim \left| a_n \right|^{1/n} \quad (\text{so } 0 \leq R \leq \infty).
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Theorem III.1.3. If \( \sum_{n=0}^{\infty} a_n(z - a)^n \), define the number \( R \) as
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Theorem III.1.3. If \( \sum_{n=0}^{\infty} a_n(z-a)^n \), define the number \( R \) as

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(c) if \( 0 < r < R \) then the series converges uniformly on \( |z-a| \leq r \).

Proof (continued). (c) Suppose \( r < R \) and choose \( \rho \) such that \( r < \rho < R \). As in the proof of (a), let \( N \in \mathbb{N} \) be such that \( |a_n| < 1/\rho^n \) for all \( n \geq N \). Then if \( |z| \leq r \), we have \( |a_n z^n| < (r/\rho)^n \) for all \( n \geq N \), and \( (r/\rho) < 1 \). Now, the Weierstrass \( M \)-Test says (Theorem II.6.2): Let \( u_n : X \to \mathbb{C} \) be a function from a metric space \( X \) to \( \mathbb{C} \) such that \( |u_n(x)| \leq M_n \) for all \( x \in X \) and suppose \( \sum_{n=1}^{\infty} M_n < \infty \). Then \( \sum_{n=1}^{\infty} u_n \) is uniformly convergent. So with \( M_n = (r/\rho)^n \), we see that the series \( \sum_{n=N}^{\infty} u_n = \sum_{n=N}^{\infty} a_n z^n \) converges uniformly on \( \{z \mid |z| \leq r\} \) (and so does \( \sum_{n=0}^{\infty} a_n z^n \)), by the Weierstrass \( M \)-Test.
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Then

(b) if \( |z - a| > R \), the series diverges, and

Proof (continued). (b) Let \( |z| > R \) and choose \( r \) with \( |z| > r > R \). Then \( 1/r < 1/R \). So, by the definition of \( \lim \), there are infinitely many \( n \in \mathbb{N} \) such that \( 1/r < |a_n|^{1/n} \). For these \( n \), \( |a_n z^n| > (|z|/r)^n \) and since \( |z|/r > 1 \), these terms are unbounded and hence the series diverges for such \( z \).
Theorem III.1.3. If \( \sum_{n=0}^{\infty} a_n(z - a)^n \), define the number \( R \) as
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\frac{1}{R} = \lim |a_n|^{1/n} \text{ (so } 0 \leq R \leq \infty) \).
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\[ (b) \text{ if } |z - a| > R, \text{ the series diverges, and } \]

Proof (continued). (b) Let \( |z| > R \) and choose \( r \) with \( |z| > r > R \). Then \( 1/r < 1/R \). So, by the definition of \( \lim \), there are infinitely many \( n \in \mathbb{N} \) such that \( 1/r < |a_n|^{1/n} \). For these \( n \), \( |a_n z^n| > (|z|/r)^n \) and since \( |z|/r > 1 \), these terms are unbounded and hence the series diverges for such \( z \).
Proposition III.1.4. If \( \sum_{n=0}^{\infty} a_n(z - a)^n \) is a given power series with radius of convergence \( R \), then \( R = \lim |a_n/a_{n+1}| \), if the limit exists.

Proof. Without loss of generality, \( a = 0 \). Let \( \alpha = \lim |a_n/a_{n+1}| \) and suppose \( |z| < r < \alpha \). Then (by the definition of limit of a sequence) there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), we have \( |a_n/a_{n+1}| > r \).
Proposition III.1.4. If $\sum_{n=0}^{\infty} a_n(z-a)^n$ is a given power series with radius of convergence $R$, then $R = \lim |a_n/a_{n+1}|$, if the limit exists.

Proof. Without loss of generality, $a = 0$. Let $\alpha = \lim |a_n/a_{n+1}|$ and suppose $|z| < r < \alpha$. Then (by the definition of limit of a sequence) there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n/a_{n+1}| > r$. Let $B = |a_N|r^N$ and then $|a_{N+1}|r^{N+1} = |a_{N+1}|rr^N < |a_N|r^N = B$ (since $|a_N| > |a_{N+1}|r$), $|a_{N+2}|rr^{N+1} < |a_{N+1}|r^{N+1} < B$, $\ldots$, and $|a_nr^n| \leq B$ for all $n \geq N$. This implies $|a_nz^n| = |a_nr^n||z^n/r^n \leq B|z^n/r^n$ for all $n \geq N$. 
Proposition III.1.4

If \( \sum_{n=0}^{\infty} a_n(z - a)^n \) is a given power series with radius of convergence \( R \), then \( R = \lim |a_n/a_{n+1}| \), if the limit exists.

**Proof.** Without loss of generality, \( a = 0 \). Let \( \alpha = \lim |a_n/a_{n+1}| \) and suppose \( |z| < r < \alpha \). Then (by the definition of limit of a sequence) there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), we have \( |a_n/a_{n+1}| > r \). Let \( B = |a_N|r^N \) and then \( |a_{N+1}|r^{N+1} = |a_{N+1}|rr^N < |a_N|r^N = B \) (since \( |a_N| > |a_{N+1}|r \) ). Let \( a_{N+1} |r^{N+1} < |a_{N+1}|r^{N+1} < B \), \( \ldots \), and \( |a_nr^n| \leq B \) for all \( n \geq N \). This implies \( |a_nz^n| = |a_nr^n| |z^n/r^n| \leq B |z^n/r^n| \) for all \( n \geq N \).

Since \( \sum_{n=1}^{\infty} B|z|^n/r^n \) is a convergent geometric series for \( |z| < r \), then by the Direct Comparison Test, the series \( \sum_{n=1}^{\infty} |a_nz^n| \) converges and the original series converges absolutely. Since \( r < \alpha \) is arbitrary, then \( \alpha \leq R \).
Proposition III.1.4 If \( \sum_{n=0}^{\infty} a_n(z-a)^n \) is a given power series with radius of convergence \( R \), then \( R = \lim |a_n/a_{n+1}| \), if the limit exists.

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Proposition III.1.4. If $\sum_{n=0}^{\infty} a_n(z - a)^n$ is a given power series with radius of convergence $R$, then $R = \lim |a_n/a_{n+1}|$, if the limit exists.

Proof (continued). Next, suppose $|z| > r > \alpha$. Then, as above, for some $N \in \mathbb{N}$, for all $n \geq N$ we have $|a_n| < r|a_{n+1}|$. Again, with $B = |a_N r^N|$, for $n \geq N$ we get $|a_n r^n| \geq B = |a_N r^N|$ and $|a_n z^n| \geq B|z|^n/r^n$ which diverges to $\infty$ as $n \to \infty$ since $|z| > r$. So $a_n z^n \not\to 0$ and by the Test for Divergence (for complex series), $\sum_{n=0}^{\infty} a_n z^n$ diverges. Since $r > \alpha$ is arbitrary, then $R \leq \alpha$. 
Proposition III.1.4. If \( \sum_{n=0}^{\infty} a_n(z - a)^n \) is a given power series with radius of convergence \( R \), then \( R = \lim |a_n/a_{n+1}| \), if the limit exists.

Proof (continued). Next, suppose \( |z| > r > \alpha \). Then, as above, for some \( N \in \mathbb{N} \), for all \( n \geq N \) we have \( |a_n| < r|a_{n+1}| \). Again, with \( B = |a_N r^N| \), for \( n \geq N \) we get \( |a_n r^n| \geq B = |a_N r^N| \) and \( |a_n z^n| \geq B|z|^n/r^n \) which diverges to \( \infty \) as \( n \to \infty \) since \( |z| > r \). So \( a_n z^n \nrightarrow 0 \) and by the Test for Divergence (for complex series), \( \sum_{n=0}^{\infty} a_n z^n \) diverges. Since \( r > \alpha \) is arbitrary, then \( R \leq \alpha \). Therefore \( R = \alpha \). \( \square \)
Proposition III.1.4. If \( \sum_{n=0}^{\infty} a_n(z - a)^n \) is a given power series with radius of convergence \( R \), then \( R = \lim |a_n/a_{n+1}| \), if the limit exists.

Proof (continued). Next, suppose \(|z| > r > \alpha\). Then, as above, for some \( N \in \mathbb{N} \), for all \( n \geq N \) we have \(|a_n| < r|a_{n+1}|\). Again, with \( B = |a_N r^N| \), for \( n \geq N \) we get \(|a_n r^n| \geq B = |a_N r^N| \) and \(|a_n z^n| \geq B|z|^n/r^n \) which diverges to \( \infty \) as \( n \to \infty \) since \(|z| > r\). So \( a_n z^n \nrightarrow 0 \) and by the Test for Divergence (for complex series), \( \sum_{n=0}^{\infty} a_n z^n \) diverges. Since \( r > \alpha \) is arbitrary, then \( R \leq \alpha \). Therefore \( R = \alpha \). \( \square \)