Proposition 3.8

Let \( z_1, z_2, \ldots, z_n \) be distinct points in the complex plane. Then the function \( f(z) = \prod_{k=1}^{n} \frac{z - z_k}{z - z_k^*} \) is analytic at each point \( z \) where \( f \) is defined, and \( f \) preserves angles at each such point.

Proof. Let \( z_1, \ldots, z_n \) be distinct points in the complex plane. Then for each \( k \), the function \( f_k(z) = \frac{z - z_k}{z - z_k^*} \) is analytic at each point \( z \) where \( f_k \) is defined. Moreover, \( f_k \) preserves angles at each such point. Thus, \( f(z) = \prod_{k=1}^{n} f_k(z) \) is analytic at each point \( z \) where \( f \) is defined, and \( f \) preserves angles at each such point.

Theorem 3.4

If \( f \) is analytic at \( z \), then \( f \) is analytic at \( z \).

Proof. Suppose \( f \) is analytic at \( z \). Then \( f \) is smooth in a region \( G \) and \( f \) is in \( G \) admits a smooth path which intersects each point \( z \) of \( G \) where \( f \) is analytic, then \( f \) preserves angles at each such point.
\[
\left\{ \begin{array}{l}
2epq + 2eq - 2pq - e2p + e2q - 2pe - 2eq
\end{array} \right.
\]

\[
\frac{|e - q|}{|e - p|} = \frac{\frac{1}{2} (q + p) + q - p + q + p}{(q - p) + (p - q)} = |q + p|
\]

\[
\Rightarrow |q + p| |q - p| = |q + p|
\]

Hence

\[
|q + p| |q - p| = (q + p)(q - p) = q^2 - p^2
\]

\[
\text{Now } \lim_{q \to \infty} g = 0 \text{ for fixed } q, \text{ implying that all such } m \text{ lie on a line.}
\]

\[
\text{Suppose } \alpha \text{ is real, then equation } (3.11) \text{ becomes}
\]

\[
\text{Case II: Suppose } \alpha \text{ is not real. Then equation } (3.11) \text{ becomes}
\]

\[
\text{Proposition III.3.10. Let } z, z', z'' \in \mathbb{C} \text{ be distinct. Then the cross ratio } (z, z', z'') 
\]

\[
= (z, z', z'') \text{ is real if and only if the four points lie on a circle/line.}
\]

\[
\boxed{\text{Proof (continued)}}
\]

\[
\text{(3.11)} \quad \alpha = \begin{cases} 0 & \text{if } \frac{p+q}{2} \not\in \mathbb{R} \\
\frac{p+q}{2} & \text{otherwise}
\end{cases}
\]

\[
\text{(m)} S = (m) S = x \text{ and so } \exists x \in \mathbb{R} \text{ such that } (z, x) \in \mathbb{R}. \text{ We know that the inverse image of } \mathbb{R} \text{ is a circle/line under any}
\]

\[
\{ (z)' T-z' z' z' \} = \{ x \in \mathbb{R} | (z) x = \} \text{ if real} \]

\[
\text{Proof. Let } \alpha : \mathbb{C} \to \mathbb{C} \text{ be defined as usual. Then the cross ratio } (z, z', z'') \text{ is real if and only if the four points lie on a circle/line.}
\]

\[
\text{Proposition III.3.10. Let } z, z', z'' \in \mathbb{C} \text{ be distinct. Then the cross ratio } (z, z', z'') 
\]

\[
= (z, z', z'') \text{ is real if and only if the four points lie on a circle/line.}
\]

\[
\text{Proposition III.3.9. } \exists z, z', z'' \in \mathbb{C} \text{ are distinct and } w, \omega \in \mathbb{C} \text{ are distinct. Then there is one and only one Mobius transformation such that}
\]

\[
\text{Proposition III.3.10. Let } z, z', z'' \in \mathbb{C} \text{ be distinct. Then the cross ratio } (z, z', z'') 
\]

\[
= (z, z', z'') \text{ is real if and only if the four points lie on a circle/line.}
\]
\[ z_1 \text{ and } z_2 \text{ are symmetric with respect to } \overline{z} = T(z). \]

\[ T(z_1, z_2, z_3, z_4) = \]

**Proposition III.3.8.**

Since \( z \) and \( z' \) are symmetric with \( L \),

\[ T(z_1, z_2, z_3, z_4) = \]

**Proposition III.3.8.**

Thus, \( z_1, z_2, z_3, z_4 \) are mapped onto \( L \),

\[ \text{LH:}\]

**Theorem III.3.19. Symmetry Principle**

That \( S \) maps \( \mathbb{C} \) one-to-one and onto \( \mathbb{C} \).

Recall \( L = \frac{\alpha - \beta}{\gamma - \beta} \).

So \( S(L) = \frac{L \cdot \gamma}{\gamma - \beta} \).

That \( S \) is real and \( \alpha, \beta, \gamma \) are real by \( S \) is real.

**Proposition III.3.19.**

For each \( z \in \mathbb{C} \), \( S(z) = \frac{z\gamma}{\gamma - \beta} \), \( z, \bar{z}, z^2, z^3, z^4 \) are real by \( S \) is real.

**Proof.**

Let \( L \) be a circle/line in \( \mathbb{C} \) and let \( S \) be a Mobius transformation takes circles/lines onto circles/lines.

**Theorem III.3.14.**

A Mobius transformation takes circles/lines onto circles/lines.

Then the cross ratio \( \{z_1, z_2, z_3, z_4\} \) is real if and only if the four points lie on a circle/line.

**Proposition III.3.10.**

Let \( z_1, z_2, z_3, z_4 \in \mathbb{C} \) be distinct. Then the cross ratio \( \{z_1, z_2, z_3, z_4\} \) is real if and only if the four points lie on a circle/line.